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Mathematical Reviews

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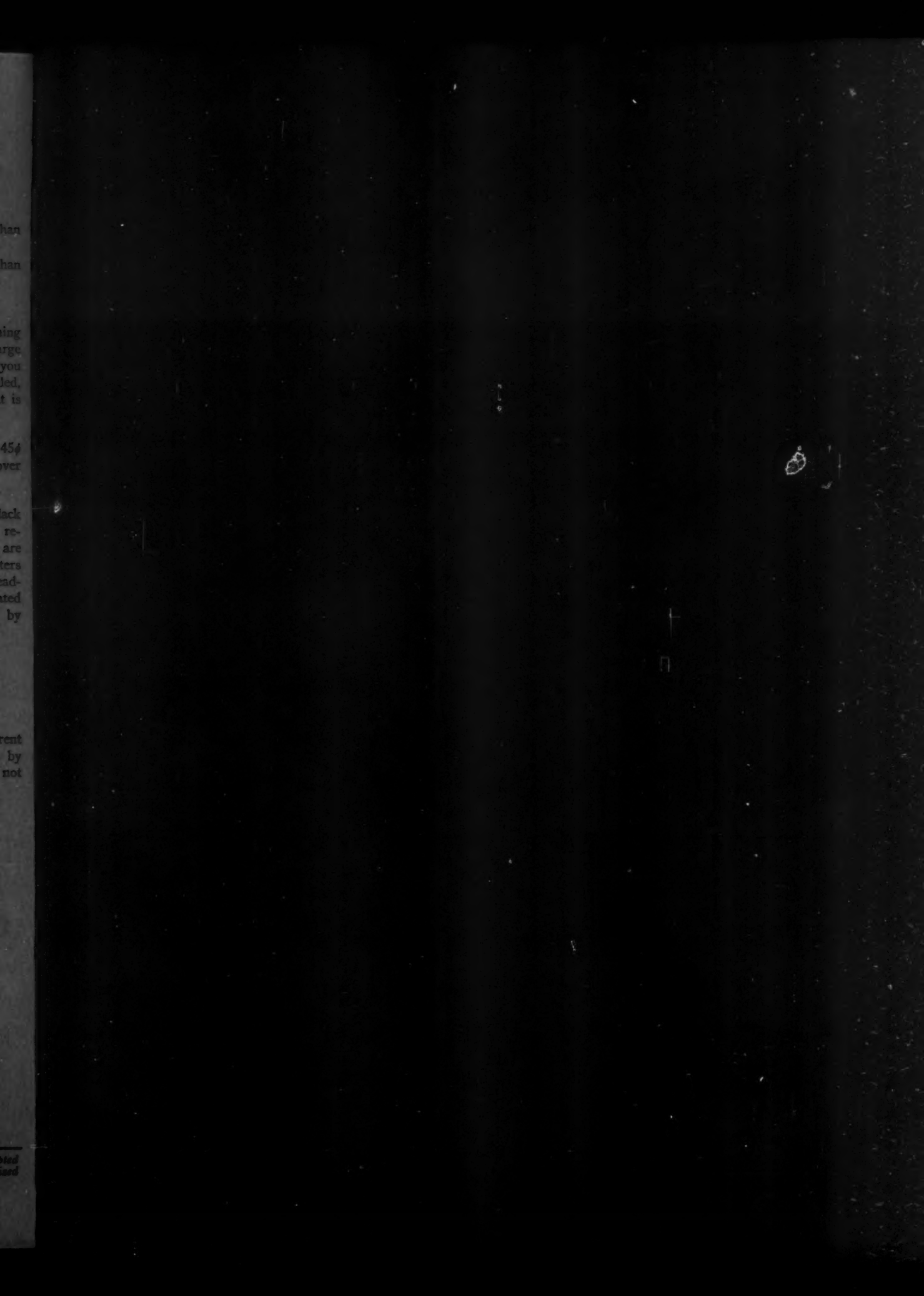

Mathematical Reviews

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ALGEBRA

Bose, R. C. A note on the resolvability of balanced incomplete designs. Sankhyā 6, 105–110 (1942). [MF 8632]

Consider an arrangement of nk objects in nr blocks of k, so that every object occurs in r of the blocks, and every pair of objects in λ of the blocks. (Therefore $\lambda(nk-1)=r(k-1)$.) This "balanced incomplete block design" is said to be resolvable if the nr blocks fall into r sets of n, such that the nk objects in each set of blocks are all different. The author proves that

$$nr \geqq nk+r-1.$$

Furthermore, the design is said to be affine resolvable if any two blocks belonging to different sets have the same number of objects in common, namely k/n. The author proves that a resolvable design is affine resolvable if and only if $nr=nk+r-1$, and that for such a design there is a nonnegative integer t such that

$$k=n(n-1)t+n, \quad r=n^2t+n+1, \quad \lambda=nt+1.$$

In the case when $n=p^\alpha$ (p prime) and $t=(n^N-1)/(n-1)$ ($N \geqq 0$), we can represent the objects and blocks as the points and hyperplanes of the finite geometry $EG(N+2, p^\alpha)$ (hence "affine"). But there are further possibilities. The case when $n=t=2$, which the author exhibits in detail, suggests the following generalization [cf. J. A. Todd, J. Math. Phys. Mass. Inst. Tech. 12, 321–333 (1933)]. When $n=2$ and $r=4t+3=p^\alpha$, we can represent the objects as the marks of $GF(p^\alpha)$ and take one pair of blocks to consist of (i) the squares, including 0, and (ii) the nonsquares and ∞. Each of the remaining pairs of blocks is derived from this initial pair by adding a definite mark to all the marks (except ∞, which remains unchanged). For instance, when $t=2$, so that $r=11$, the initial pair is (0, 1, 3, 4, 5, 9), (2, 6, 7, 8, 10, ∞), another pair is (1, 2, 4, 5, 6, 10), (3, 7, 8, 9, 0, ∞), and so on, making eleven pairs in all.

H. S. M. Coxeter (Toronto, Ont.).

Aitken, A. C. The monomial expansion of determinantal symmetric functions. Proc. Roy. Soc. Edinburgh. Sect. A. 61, 300–310 (1943). [MF 8301]

It was shown by Jacobi that the quotient of alternants

$$|\alpha^s\beta^{r+1}\gamma^{q+2}\cdots|/|\alpha^s\beta^r\gamma^{q+1}\cdots|, \quad 0 \leqq p \leqq q \leqq \cdots,$$

is expressible as an isobaric determinant whose elements are the complete homogeneous symmetric functions h_r in the variables $\alpha, \beta, \gamma, \cdots$, the weight of any element being one less than that of the element directly above it, and having $h_p, h_q, h_r, \cdots$ as principal diagonal. The expansion of this determinant in monomial terms was given by Littlewood [Proc. London Math. Soc. (2) 43, 226–240 (1937)]. In this note the author's object is "to establish by elementary reasoning the monomial expansion of the general isobaric h-determinant, in which the steps of suffixes from one row to the next can be any integers." The theorem obtained is expressed in terms of the notion of associated asterisk diagrams, a device introduced by the author in an earlier paper [Proc. Edinburgh Math. Soc. (2) 1, 51–61 (1927)]. *L. M. Blumenthal* (Columbia, Mo.).

MacDuffee, C. C. What is a matrix? Amer. Math. Monthly 50, 360–365 (1943). [MF 8487]

Expository article.

Scott, Winston M. A remark on algebras of matrices. Bull. Amer. Math. Soc. 49, 444–446 (1943). [MF 8394]

Let $\mathfrak{A}$ be a matrix algebra, with unit element, over an algebraically closed field K. A matrix module with $\mathfrak{A}$ as both right and left operator system is called an $(\mathfrak{A}, \mathfrak{A})$ module. By applying a theorem of R. Brauer [Trans. Amer. Math. Soc. 49, 502–548 (1941); these Rev. 3, 34] the author proves that a composition series $\mathfrak{A}=\mathfrak{A}_0 \supset \mathfrak{A}_1 \supset \mathfrak{A}_2 \supset \mathfrak{A}_3 \supset \cdots \supset \mathfrak{A}_m$ of a matrix algebra considered as an $(\mathfrak{A}, \mathfrak{A})$ module is also a composition series of $\mathfrak{A}$ considered as an $(\mathfrak{A}^*, \mathfrak{A}^*)$ module, where $\mathfrak{A}$ is considered as the direct sum of $\mathfrak{A}^*$, a semisimple subalgebra, and $\mathfrak{R}$, the radical of $\mathfrak{A}$. An application of this result yields an indirect proof of a theorem previously obtained by the author. [Ann. of Math. (2) 43, 147–160 (1942); these Rev. 3, 263]. *N. H. McCoy.*

Dines, Lloyd L. On linear combinations of quadratic forms. Bull. Amer. Math. Soc. 49, 388–393 (1943). [MF 8384]

Let $Q=a_{ik}x_ix_k$, $B=b_{ik}x_ix_k$, $(Q, B)=a_{ik}b_{ik}$, where $a_{ik}=a_{ki}$, $b_{ik}=b_{ki}$. If $(Q, B)=0$, then Q and B will be said to be c-orthogonal. The present paper is concerned principally with the proof of the following three statements in regard to a linear combination $Q=\lambda_iQ_i$ of m quadratic forms: (a) Q can be chosen to be definite if and only if every quadratic form c-orthogonal to all the Q_i is indefinite; (b) Q is indefinite or identically zero if there exists a definite quadratic form c-orthogonal to all the Q_i; (c) Q can be chosen to be semi-definite (but not definite) if and only if there exists a semi-definite (but no definite) quadratic form c-orthogonal to all the Q_i. *M. R. Hestenes* (Chicago, Ill.).

Arf, Cahit. Untersuchungen über quadratische Formen in Körpern der Charakteristik 2. I. J. Reine Angew. Math. 183, 148–167 (1941). [MF 8646]

Let $F=\sum_{1 \leqq i \leqq j \leqq n} a_{ij}x_ix_j$ be a quadratic form over a field k of characteristic 2. To study the equivalence of forms the author interprets F as the square of the norm of a vector in an n-dimensional vector space $\mathfrak{R}$ over k; similarly the bilinear form of F is considered to be the inner product on $\mathfrak{R}$. Then two forms F and F' are equivalent over k if and only if the associated vector spaces $\mathfrak{R}$ and $\mathfrak{R}'$ are isometric. Let $\mathfrak{R}^*$ be the invariant subspace of $\mathfrak{R}$ consisting of all $\mathfrak{x}^*$ with $(\mathfrak{x}^*, \mathfrak{y})=0$ for all $\mathfrak{y}$ in $\mathfrak{R}$. Then $\mathfrak{R}$ is the direct sum $\sum_{i=1}^{s}\langle \mathfrak{u}_i, \mathfrak{v}_i\rangle+\mathfrak{R}^*$, where $\dim\langle \mathfrak{u}_i, \mathfrak{v}_i\rangle=2$, $\mathfrak{u}_i, \mathfrak{v}_i \neq 0$ and $(\mathfrak{u}_i, \mathfrak{v}_i)$ has no subspace $\mathfrak{R}^*$; that is, they are completely regular.

Passage to the quadratic forms then results in the equivalence

$$F\sim\sum_{i=1}^{n}(a_i x_i^2+b_i x_i y_i+c_i y_i^2)+\sum_{j=1}^{n^*} d_j z_j^2=\sum f_i+F^*,\quad b_i\neq 0.$$

The form F^* is uniquely determined to within equivalence transformations. Similarly, the number $2n$ of the totally regular part $\sum f_i$ is unique, though $\sum f_i$ is not. To obtain further invariants the author studies the Clifford algebra $C(F)$ with $u_i^2=a_{ii}$, $u_iu_j+u_ju_i=a_{ij}$, $i<j$. Then $C(F)=\prod C(f_i)C(F^*)$. Reducing $C(F)$ modulo the radical, one finds that only $\prod C(f_i)$ and thus $C(f_i)$ have to be investigated. For totally regular forms $\sum f_i$ the algebra $\prod C(f_i)$ and residue class $\Delta(F)\equiv\sum(a_ic_i)/b_i^2 \pmod{a^2-a,\ a \text{ in } k}$ are invariants. To obtain complete sets of invariants the author studies the necessary and sufficient conditions which describe whether a form represents 0. These conditions depend on the number of variables. A typical result is this: "If each form $f_1+f_2+z^2$ represents 0 in k then the number of variables $2n$, $C(F)$ and $\Delta(F)$ form a complete set of invariants for equivalence." The author is unaware of the work of A. A. Albert [Ann. of Math. (2) **39**, 494–505 (1938); Trans. Amer. Math. Soc. **43**, 386–436 (1938)].

O. F. G. Schilling (Chicago, Ill.).

Abstract Algebra

Braithwaite, R. B. Characterisations of finite Boolean lattices and related algebras. J. London Math. Soc. **17**, 180–192 (1942). [MF 8276]

M. H. A. Newman's axiomatic characterization of Boolean algebras (Boolean lattices) [J. London Math. Soc. **16**, 256–272 (1941); **17**, 34–47 (1942); these Rev. **4**, 70] is pushed further and modified. As in the work of Newman, commutativity and associativity are dispensed with in favor of distributivity, idempotence, and properties of complements. The postulates for Boolean algebras whose adequacy is demonstrated by Braithwaite are essentially: (i) distributivity $a(b\cup c)=ab\cup ac$ and $(b\cup c)a=ba\cup ca$. (ii) there exists a left-zero 0 such that $0\cup a=a$ for all a and such that $a\cup b=0$ implies $a=0$, (iii) for every a there exists an a' such that $a\cap a'=0$ and such that, except for one $a=I$, $a'\neq 0$, (iv) given a, we can find $a_2,\cdots,a_m$ such that $a_k(a_{k-1}\cup(a_{k-2}\cup\cdots(a_2\cup a)\cdots))=0$ for $k=2,\cdots,m$, and such that $b(a_m\cup\cdots(a_2\cup a))=0$ implies $b=0$ (that is, $a_m\cup\cdots(a_2\cup a)=I$), and (v) $aa=a$ for all a. We can replace (v) by (v′) $Ia=a$ for all a.

G. Birkhoff.

Kurosh, A. Direct decompositions of simple rings. Rec. Math. [Mat. Sbornik] N.S. **11(53)**, 245–264 (1942). (English. Russian summary) [MF 8370]

The rings discussed in this paper are simple rings with unit, that is, simple algebras of possibly infinite order. In carrying out the generalization several new ideas are introduced. If A is a proper left ideal in the ring R, the maximal subring A^* of R in which A is a two-sided ideal is called the "eigenring" of A, and A^*/A is called the ring "subjected to R by means of A." If R is normal simple with center P, R is "fundamental" if P is among the rings subjected to R. The role played by fundamental rings is indicated by the theorem that the direct product of a normal simple ring and its inverse is fundamental. For algebras of finite order the fundamental rings are precisely matric algebras. A ring R, with field of operators P, is locally simple (normal, matricial) if every finite subset of R is contained in a simple (normal, matricial) algebra of finite order over P. A locally matricial ring with denumerable basis is shown to be a direct product of matric algebras, but examples show that this fails for nondenumerable bases.

In conclusion the author considers the classical result that, if A is a finite normal simple subalgebra of R, then A is a direct factor of R; he shows that, conversely, if A is a direct factor of every ring in which it is embedded, A must be normal simple. Whether finite order of A is also implied remains in doubt, but an example is given of a normal simple (even locally matricial) ring for which the theorem in question fails.

I. Kaplansky (Cambridge, Mass.).

Levitzki, Jakob. On the radical of a general ring. Bull. Amer. Math. Soc. **49**, 462–466 (1943). [MF 8400]

The radical N of a ring S has been variously defined as the union of all nilpotent ideals or of all nil-ideals. The author proposes to call these the specialized radical and generalized radical, respectively, and reserves the term radical for the union of all semi-nilpotent ideals, an ideal being semi-nilpotent if any subring generated by a finite number of its elements is nilpotent. (For algebras these three types of radical coincide). With his definition the author succeeds in generalizing the result that S/N has radical zero. If further S/N is semi-simple, S is an "A-ring"; this is equivalent to the minimal condition on the right ideals containing N. A ring with unit all of whose proper ideals (right ideals) are semi-nilpotent is called primary (completely primary); the author's final result, a generalization of the Wedderburn theorem, states that a primary A-ring is a ring of matrices over a completely primary ring.

I. Kaplansky (Cambridge, Mass.).

Bruck, R. H. Addendum to the paper "Generalized Fischer groups and algebras". Bull. Amer. Math. Soc. **49**, 461 (1943). [MF 8399]

The paper appeared in the same Bull. **48**, 618–626 (1942); these Rev. **4**, 71.

Lee, Herbert Leonard. Power sums of polynomials in a Galois field. Duke Math. J. **10**, 277–292 (1943). [MF 8468]

The author evaluates certain sums of powers of polynomials with coefficients in the Galois field $GF(p^n)$: $S_m{}^k=\sum' M^k$, $\sigma_m{}^k=\sum' M^{-k}$, where M ranges over all monic polynomials of degree m, and similar sums $R_m{}^k$, $\rho_m{}^k$, where the range is all polynomials of degree less than m. Write

$$k=a_sp^{ns}+\cdots+a_1p^n+a_0,\quad 0\leqq a_i\leqq p^n-1,$$

and $\mu(k)=\sum a_i$. For $\mu(k)<m(p^n-1)$, $R=S=0$; and in general, for $s\leqq m$ the sums are particularly simple and can be given in a single term. The case $s>m$ is handled by recursion formulae, but explicit evaluations are given for certain special forms of k, for example, when $k+1$ is a sum of powers of p^n. In conclusion the author exhibits a connection between the sums R and ρ and the complete symmetric homogeneous polynomial of degree k.

I. Kaplansky.

Wagner, R. W. Differentials and analytic continuation in non-commutative algebras. Duke Math. J. **9**, 677–691 (1942). [MF 7922]

The author considers functions in an arbitrary algebra $\mathfrak{A}$ over the real or the complex field. By using the regular representation he extends the absolute value of matrices defined by Rella to an absolute value for elements in $\mathfrak{A}$. This absolute value is used to define the metric in $\mathfrak{A}$. The author defines a generalized power series $F(X)$ to be a function of the form $\sum G_k(X)$, where

$$G_k(X)=\sum A_0XA_1\cdots XA_k.$$

An important tool in this study is the Fréchet differential defined to be a function $D(F(X), H)$ which is linear and homogeneous in H and has the property that for any $e>0$ there is a $d>0$ such that if $|H|\leqq d$ then

$$|F(X+H)-F(X)-D(F(X), H)|\leqq e|H|.$$

The author proves that, if $X=\sum x_iE_i$ and $Y=F(X)=\sum y_iE_i$, where the E's form a basis for $\mathfrak{A}$, then necessary and sufficient conditions that $F(X)$ have a generalized power series are that each y_s have a power series in the x_s and that, for all n, $D^n[F(X), H]$ exists and is expressible as a homogeneous polynomial in H with multipliers depending on X. Here $D^n[F(X), H]$ is the nth differential defined by Graves. If $F(X)$ is defined by a generalized power series which converges uniformly and absolutely on $|X|\leqq 2c$, then, in $|X|<c$, $|H|<c$,

$$F(X+H)=\sum D^r[F(X), H]/r!.$$

This theorem is useful in obtaining analytic continuations of functions defined by generalized series. The author considers also functions defined by series of the form $\sum g^{(r)}(\lambda_i)P_iQ_i/r!$, where the λ_i are the characteristic roots of X and the P_i and Q_i form a set of partial idempotent elements. *N. Jacobson* (Chapel Hill, N. C.).

Hasse, Helmut. Der n-Teilungskörper eines abstrakten elliptischen Funktionenkörpers als Klassenkörper, nebst Anwendung auf den Mordell-Weilschen Endlichkeitssatz. Math. Z. **48**, 48–66 (1942). [MF 8561]

The author has distilled an abstract essence from Mordell's proof of his well-known theorem [L. J. Mordell, Proc. Cambridge Philos. Soc. **21**, 179–192 (1922)], according to which the rational points on a curve of genus 1 form an Abelian group with a finite basis. His work is based on the version given by Weil of that proof [A. Weil, Bull. Sci. Math. (2) **54**, 182–191 (1930)]. Mordell's proof consisted of two parts, the first one of which studies the arithmetical properties of the bisection of elliptic functions, while the other applies these to an "infinite descent." The main part of the present paper deals with the first step; the purely algebraic results, which are implicit in Mordell and were in part made explicit by Weil [loc. cit., and Acta Math. **52**, 281–315 (1928)], are here developed, in the language peculiar to the author and his school, for an elliptic function-field over an arbitrary (perfect) field of constants, and the division by n. The main result is stated as a theorem on the decomposition of a prime divisor of degree 1, belonging to the given function-field, in the Abelian extension of degree n^2 determined by the division by n (the divisors of order n being assumed to be in the field of constants). As pointed out by the author, there is an interesting analogy between this and the law of reciprocity in ordinary classfield theory. The author states that this has its source in the classical theorem of the "exchange of parameter and argument" for an Abelian integral of the third kind; it seems to the reviewer that this is not quite correct; the source of the result in question is rather to be looked for in Cauchy's theorem applied to the integral $\int \log \varphi \cdot d(\log \psi)$, where φ and ψ are in the function-field; all the available evidence points to that integral as being the correct analogue of the law of reciprocity. The final section of the paper deals with the "infinite descent" in the case of an algebraic number-field; this is here carried out on the basis of the so-called distribution-theory, and essentially in the same way as had been done by Mordell and Weil, except that a different birational model is used for the curve. *A. Weil.*

Hasse, Helmut. Zur arithmetischen Theorie der algebraischen Funktionenkörper. Jber. Deutsch. Math. Verein. **52**, 1–48 (1942). [MF 8531]

This, the author states, is the introductory section of a report, the writing of which was interrupted. The report was to cover the following topics, which the author lists as the most important results hitherto achieved in the study of algebraic curves over arithmetically significant fields of constants: (I) Weil's theorem of the finite basis [A. Weil, Acta Math. **52**, 281–315 (1928)]; (II) Siegel's theorem on the finite number of integral solutions [C. L. Siegel, Abh. Preuss. Akad. Wiss. Phys.-Math. Kl. **1929**, part 2]; (III) Lutz's theorem on elliptic curves over p-adic fields [E. Lutz, J. Reine Angew. Math. **177**, 238–247 (1937)]; (IV) Hasse's proof of the Riemann hypothesis for elliptic curves over Galois-fields [H. Hasse, J. Reine Angew. Math. **175**, 55–62, 69–88, 193–208 (1936)] [this has now been superseded by the proof obtained for arbitrary genus by A. Weil, Proc. Nat. Acad. Sci. U. S. A. **27**, 345–347 (1941); these Rev. **2**, 345]; (V) Deuring's algebraic construction of the Abelian extensions of the function-field of an algebraic curve [M. Deuring, Math. Ann. **106**, 77–102 (1932)]. These topics were to be treated from a "unified point of view," which, needless to say, is supplied by the consideration of the Jacobian variety (or, in the language of the author, the Abelian function-field) belonging to the curve. The greater part of the present paper is devoted to an exposition of some of the more elementary properties of the Jacobian variety, in purely algebraic terms and without the use of the convenient tools supplied in the case of characteristic 0 by classical function-theory. These properties are of course couched in the arithmetico-algebraic language of the author and his school, which will be familiar to readers of his papers on elliptic function-fields, but may act as a deterrent on other classes of readers, and does not seem to the reviewer to be as well adapted to those questions as the language of algebraic geometry. Some new lemmas, mainly of technical interest, are stated and proved; some of the basic results of the classical theory are stated as unproved assertions in the abstract case. A final section is devoted to an exposition, restricted to algebraic curves, of Weil's so-called distribution-theory [loc. cit. chap. I], with the quantitative complements due to Siegel [loc. cit.]. One may express the hope that circumstances will soon restore the author to his mathematical studies and enable him to complete his report. *A. Weil* (Bethlehem, Pa.).

NUMBER THEORY

Bell, E. T. Note on a conjecture due to Euler. Bull. Amer. Math. Soc. 49, 393–394 (1943). [MF 8385]

The conjecture referred to is to the effect that the equation

$$x_1^n+x_2^n+\cdots+x_t^n=x^n$$

has no solution $(x_1, \cdots, x_t, x)$ in rational numbers (different from zero) in case $n>3$ and $2<t<n$. The author points out that this equation has solutions in algebraic numbers for $n>3$ and for any $t>1$. This solution is (to within a factor of homogeneity) as follows:

$$\begin{aligned} x_1&=1,\\ x_2&=r_1,\\ x_3&=r_2(1+r_1),\\ x_4&=r_3(1+r_2)(1+r_1),\\ &\cdots\cdots\cdots\\ x_t&=r_{t-1}(1+r_{t-2})\cdots(1+r_1),\\ x&=(1+r_{t-1})\cdots(1+r_1), \end{aligned}$$

where the r's are chosen in any way from among those roots of $(1+x)^n=1+x^n$ which are different from 0 and -1.

D. H. Lehmer (Berkeley, Calif.).

Niven, Ivan. The Pell equation in quadratic fields. Bull. Amer. Math. Soc. 49, 413–416 (1943). [MF 8389]

The author [Trans. Amer. Math. Soc. 52, 1–11 (1942); these Rev. 4, 34] has shown that, if γ is an integer of a quadratic field F, then the equation $\xi^2-\gamma\eta^2=1$ has an infinite number of integral solutions in F if and only if γ is not totally negative when F is a real field, and γ is not the square of an integer of F when F is imaginary. Here he considers the possibility of representing the solutions in the form

$$\begin{aligned} \xi&=\{(\xi_1+\gamma^{\frac12}\eta_1)^n+(\xi_1-\gamma^{\frac12}\eta_1)^n\}/2,\\ \eta&=\{(\xi_1+\gamma^{\frac12}\eta_1)^n-(\xi_1-\gamma^{\frac12}\eta_1)^n\}/(2\gamma^{\frac12}), \end{aligned}$$

where ξ_1, η_1 is, in a sense, a minimal solution. The result is that such a representation can be found whenever the equation has an infinite number of solutions if and only if γ is not a totally positive nonsquare integer of F. In the statement of the theorem the value $n=0$ should be allowed in order to include the trivial solution $\xi=1$, $\eta=0$.

H. S. Zuckerman (Seattle, Wash.).

Pillai, S. S. On $\sigma_{-1}(n)$ and $\phi(n)$. Proc. Indian Acad. Sci., Sect. A. 17, 67–70 (1943). [MF 8447]

Results on the orders of $\sigma_{-1}(n)$ and $\phi(n)$ due to Wigert [Acta Math. 37, 113–140 (1914)] and Landau [Handbuch der Verteilung der Primzahlen, Leipzig, 1909, p. 216] are obtained with certain estimates of error terms.

H. S. Zuckerman (Seattle, Wash.).

Sambasiva Rao, K. and Nagabhushanam, K. A note on Vinogradow-numbers. Proc. Indian Acad. Sci., Sect. A. 17, 71–74 (1943). [MF 8448]

Let $S(\alpha, N)=\sum e(\alpha p)$, where the summation is over all primes $p\leqq N$. The authors define a Vinogradow number as follows: a number α such that $0<\alpha\leqq 1$ is a Vinogradow number if for every $h>1$ there is a $c=c(h)$ for which $|S(\alpha, N)|<cN(\log N)^{-h}$ for all N. They prove, among other results, that the set of Vinogradow numbers has measure 1.

R. D. James (Saskatoon, Sask.).

Benneton, Gaston. Sur un problème d'Euler. C. R. Acad. Sci. Paris 214, 459–461 (1942). [MF 7859]

The four rows of a rational orthogonal matrix of order 4 can, after permutations, sign-changes and multiplying up by a common denominator, be expressed by $UJ_\alpha V$ ($\alpha=0, 1, 2, 3$), where the J_α are the quaternion units and U, V are integral quaternions. Euler gave a numerical example containing 16 different coefficients, with norm (sum of squares) 8515. Similar examples are given here, of least norm 354, and of norms 498 and 7150. The ternary example of least norm, with nine distinct rational direction cosines, has the rows (52, 23, 4)/57, (17, −44, 32)/57, (16, −28, −47)/57.

G. Pall (Montreal, Que.).

Benneton, Gaston. Sur l'arithmétique des quaternions. C. R. Acad. Sci. 214, 406–408 (1942). [MF 7850]

Conditions are stated for integral quaternions to have the same left or right divisors of a prime norm p. The norm d of the greatest common left (or right) divisor of two quaternions A and B is the largest rational integer dividing simultaneously $\bar{A}B$ (or $A\bar{B}$) and the norms of A and B. Remarks are made on the simultaneous representations $A=UA_1V$ and $B=UB_1V$, where A_1 and B_1 have no common left or right divisors. Most of these results can be deduced from an article by the reviewer [Trans. Amer. Math. Soc. 47, 487–500 (1940); these Rev. 2, 36].

G. Pall.

Pall, Gordon. The distribution of integers represented by binary quadratic forms. Bull. Amer. Math. Soc. 49, 447–449 (1943). [MF 8395]

In this paper the author proves that the number $c(x)$ of positive integers $n\leqq x$ which can be represented by positive primitive binary quadratic forms of a given negative discriminant d is given by

$$cx/(\log x)^{\frac12}+O(x/\log x),$$

where c is positive and depends on d alone. This result is deduced from the theorem of R. D. James [Amer. J. Math. 60, 737–744 (1938)] which states that the above formula with a different constant in place of c gives the number $B(x)$ of all those $n\leqq x$ in the above, for which $(n, d)=1$.

A. E. Ross (St. Louis, Mo.).

Tchudakoff, N. On Siegel's theorem. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 6, 135–142 (1942). (Russian. English summary) [MF 7692]

This paper gives a modification of Heilbronn's simplification [Quart. J. Math., Oxford Ser. 9, 194–195 (1938)] of Siegel's proof that $hR\geqq k^{\frac12-\epsilon}$ if $k\geqq k_0(\epsilon)$, where h is the number of binary quadratic classes of determinant $\pm k$ and R the regulator. Let $f(s)$ denote an analytic function defined by $\sum a_n n^{-s}$, convergent in a half-plane, such that $f(s)$ is meromorphic and $f(s)=O(|t|^\beta)$ uniformly over the half-plane $\sigma\geqq\frac12$ excepting the poles, and such that for certain positive constants λ and A, and an integer q, $A^s\Gamma^q(\lambda s)f(s)=F(s)$ satisfies the functional equation $F(1-s)=F(s)$. If $F_0(s)$ is the rational function part of $F(s)$, containing all the poles, if $\mu=(q\lambda)^{-1}$, and Nx and Sx stand for $x_1\cdots x_q$ and $x_1+\cdots+x_q$, then

$$(1)\quad F(s)-F_0(s)=\sum_{n=1}^{\infty} a_n \int\cdots\int_{Nx\geqq 1}[(Nx)^{\lambda s-1}+(Nx)^{\gamma(1-s)-1}]e^{-A^{-q}n^\mu Sx}dx_1\cdots dx_q.$$

This is applied with $f(s)=\zeta(s)L(s,\chi)L(s,\chi_1)L(s,\chi_2)$, where the χ's are real characters for which the a_n turn out to be nonnegative, so that the inequality obtained by dropping all but the first term on the right of (1) suffices to prove $1-\rho>k^{-\epsilon}$ if $k\geqq k_0(\epsilon)$, where ρ is the largest real root of $L(s,\chi)$, χ being a character mod k. No use is made of algebraic numbers. *G. Pall* (Montreal, Que.).

Stöhr, Alfred. Bemerkungen zur additiven Zahlentheorie. I. J. Reine Angew. Math. **183**, 168–174 (1941). [MF 8647]

Let A be any set of natural numbers including the number 1 and $l(m)$ the smallest number such that m is the sum of $l(m)$ numbers of A. Define

$$h_A=\max_{m=1,2,\cdots} l(m)$$

when it exists and

$$\lambda_A=\sup_{x=1,2,\cdots}(1/x)\sum_{m=1}^{x} l(m)$$

when it exists. Among the results of the author are the following. Theorem 1. If λ_A exists, then h_A exists and $h_A<2\lambda_A$. Theorem 2. If h_A exists and $l\geqq h_A$, then

$$\lambda_A\leqq l-\sum_{n=1}^{l-1}\alpha_n,$$

where α_n is the Schnirelman density of nA. A sufficient condition for the equality to hold in theorem 2 is found as well as a necessary and sufficient condition that $\lambda_A=h_A$. *B. W. Jones* (Ithaca, N. Y.).

Linnik, U. V. On a sequence which does not form the binary basis. C. R. (Doklady) Acad. Sci. URSS (N.S.) **36**, 163–165 (1942). [MF 8004]

A sequence of natural members $F=\{M_j\}$ is a binary basis if $F+F$ represents all large numbers. A sequence is constructed such that $F+F+F$ represents all large numbers and $F+F$ only almost all. Let N_0 be sufficiently large, $N_k=N_0^{4^k}$, and choose primes p_k ($\equiv 3$, mod 4) such that p_k is in the interval $[.999N_k^{1/5}, 1.001N_k^{1/5}]$. Let Δ_k denote the interval $[N_k^{1/4}, N_k]$, $k=0, 1, \cdots$. Form a sequence $\{L_j\}$ consisting of all integers in Δ_0, but only of the quadratic residues mod p_k in each Δ_k ($k\geqq 1$); hence if $p_k n$ is in the interval $\eta_k=[N_k/3, 2N_k/3]$ and $L_{j_1}\leqq L_{j_2}$, $L_{j_1}+L_{j_2}=p_k n$ is impossible unless $L_{j_1}\leqq N_k^{1/4}$. In η_k choose $[N_k^{1/10}]$ numbers $p_k n$, and delete all numbers L_{j_2} from the sequence; their number does not exceed $N_k^{1/4+1/10}$. The sequence remaining is F. It is shown to have the properties: the number $Q(N)$ of M_j in $[1, N]$ exceeds $\frac{1}{4}N$; the difference between consecutive M_j is $O(N^{.45})$; every arithmetic progression $qx+l$, $2\leqq q\leqq N^{1/1000}$, contains $Q/q+O(Q/qN^{1/2000})$ numbers on $[1, N]$; $|\sum_{j=1}^{Q(N)}\exp 2\pi i\alpha M_j|<Q/N^{1/1000}$ if α is in $[1/N^{.55}, 1-1/N^{.55}]$, and $\ll 1/(\alpha)$ otherwise; yet there exists an infinite sequence of intervals $[N_k/3, 2N_k/3]$ each containing no less than $N_k^{1/10}$ numbers not represented by $F+F$. *G. Pall* (Montreal, Que.).

Gupta, Hansraj. An inequality in partitions. J. Univ. Bombay **11**, 16–18 (1942). [MF 8305]

Let $p_k(n)$ denote the number of partitions of n into precisely k positive integers. By use of the identity

$$p_k(n)=p_{k-1}(n-1)+p_{k-1}(n-k-1)+p_{k-1}(n-2k-1)+\cdots$$

an inductive proof is given of the inequality

$$\binom{m+a_{k-1}}{k-1}\leqq k!\,p_k(m+1)\leqq\binom{m+a_k}{k-1},$$

where $a_k=k(k-1)/4$ and where $k\geqq 4$ and m is sufficiently large depending on k. The argument shows that, as $n\to\infty$,

$$p_k(n)\sim\binom{n-1+a_k}{k-1}\Big/k!.$$

D. H. Lehmer (Berkeley, Calif.).

ANALYSIS

Theory of Functions of Complex Variables

Vicente Gonçalves, J. Contours de Jordan et intégrale de Cauchy. Portugaliae Math. **3**, 253–257 (1942). [MF 8515]

This paper supplies additional details of proof to those appearing in a previous paper under the same title [Portugaliae Math. **2**, 166–172 (1941); these Rev. **3**, 76]. The occasion for this paper is the author's defense of the validity of his results which were questioned in another abstract [Zentralblatt Math. **25**, 151.] *D. Moskovitz.*

Kalustyan, H. Représentation conforme et mouvement d'un plan sur un plan. Rev. Fac. Sci. Univ. Istanbul Ser. A. **6**, 135–143 (1941). (French. Turkish summary) [MF 8452]

For an analytic function $Z=f(z)$, with curve c in the z-domain of definition mapped on curve C in the Z-domain, the author determines the correspondence of differential elements of higher order, or elements of curvature, analogous to the familiar correspondence of tangents and of elements of arc. Application is made to the determination of the successive evolutes of C in terms of $f(z)$ and the equation of c. *E. F. Beckenbach* (Austin, Tex.).

Sunyer Balaguer, F. On some results concerning the theorems of Picard, Landau and Schottky and on a criterion of quasinormality. Revista Mat. Hisp.-Amer. (4) **2**, 88–96, 271–278 (1942). (Spanish) [MF 7157]

The author deduces from a result of R. Nevanlinna [Eindeutige Analytische Funktionen, Springer, Berlin, 1936, chap. 9] that for a meromorphic function, not a constant, it is impossible to have for all large r

$$(*)\qquad n(r, z_0)\geqq(1+\alpha)[n(r, z_1)+n(r, z_2)+n(r, z_3)],$$

where z_0, z_1, z_2, z_3 are distinct and $\alpha>0$. He then proves a criterion for quasi-normality of a family of meromorphic functions and extensions of Landau's theorem and Schottky's theorem suggested by this result; in each case an inequality of the form (*) is involved. *E. S. Pondiczery* (Princeton, N. J.).

Robinson, Raphael M. Analytic functions in circular rings. Duke Math. J. **10**, 341–354 (1943). [MF 8473]

The following problem is solved. Suppose that $f(z)$ is meromorphic and single-valued for $q\leqq|z|\leqq 1$ and that $|f(z)|\leqq p$ for $|z|=q$ and that $|f(z)|\leqq 1$ for $|z|=1$. Suppose further that

$$f(z)\frac{(z-b_1)\cdots(z-b_k)}{(z-a_1)\cdots(z-a_k)}$$

can be defined to be analytic throughout $q \leqq |z| \leqq 1$, where the a's and b's are given interior points of the ring. If z_0 is a given interior point of the ring, what is the largest possible value for $|f(z_0)|$?

The solution of this problem is based upon the following lemma which is noteworthy for its power and simplicity. If $f(z)$ is regular for $q \leqq |z| \leqq 1$, except perhaps for a simple pole at $-b$ $(q<b<1)$, and if $|f(z)| \leqq 1$ for $|z|=q$ and for $|z|=1$, then $|f(z)| \leqq 1$ for $q<z<1$. If the equality holds for any point of this interval, then $f(z)$ is a constant. This lemma is concluded from the theorem: if $f(z)$ is regular within and on the boundary of a region except for a simple pole inside, and if $|f(z)| \leqq 1$ on the boundary, then $f(z)$ assumes every value of $|w|>1$ exactly once in the region. With the aid of the lemma the extremal value for $|f(z_0)|$ and the corresponding extremal functions are found.

The lemma is also applied to the problem of improving Hadamard's three circles theorem for functions which are single-valued and analytic in a ring [cf. the paper by the reviewer, Amer. J. Math. 62, 91–106 (1940); these Rev. 1, 114; cf. also Teichmüller, Deutsche Math. 4, 16–22 (1939)]. The normalized extremal function is determined first by the aid of the theory of the conformal mapping of doubly-connected regions and is then independently represented analytically in terms of theta functions. The correction factor for Hadamard's theorem is computed in terms of the modulus of the Jacobi theory of elliptic functions and a simple numerical case is considered. By specializing the solution of the problem first proposed the author establishes the analogue of Schwarz's lemma for a circular ring.

M. H. Heins (Chicago, Ill.).

Kober, H. A note on approximation by rational functions. Bull. Amer. Math. Soc. 49, 437–443 (1943). [MF 8393]

This paper considers L^p approximation by rational functions either on the real axis or in the upper half plane. A function belonging to $L^p(-\infty, \infty)$ $(0<p \leqq \infty)$ can be approximated in the L^p metric by rational functions whose only poles are at two assigned points, one in the upper half-plane and one in the lower half-plane. The function can be approximated in the same metric by rational functions with poles only at one point in the lower half-plane if and only if it is equivalent to the limit function of a function $F(z)$ belonging to the Hille-Tamarkin class $\mathfrak{H}_p$ in the upper half-plane; in this case, $F(z)$ can be L^p approximated in the upper half-plane by rational functions with poles at the same point. The cases $p=\infty$ are contained in results of Walsh [Interpolation and Approximation by Rational Functions in the Complex Domain, Amer. Math. Soc. Colloquium Ser., v. 20, New York, 1935, p. 47]. For $p=1, 2, \infty$, explicit approximating functions are given, and the theory is illustrated by application to the function

$$\tfrac{1}{2}s(1-s)\Gamma(\tfrac{1}{2}s)\pi^{-s/2}\zeta(s).$$

R. P. Boas, Jr. (Cambridge, Mass.).

Siegel, Carl Ludwig. Symplectic geometry. Amer. J. Math. 65, 1–86 (1943). [MF 7778]

Siegel has introduced modular functions of degree n, that is, of $m=\frac{1}{2}n(n+1)$ complex variables [Math. Ann. 116, 617–657 (1939); these Rev. 1, 203]. Similarly generalized automorphic functions can be defined [Sugawara, Ann. of Math. (2) 41, 488–494 (1940); these Rev. 2, 37]. Such a theory is based on (1) a suitable definition of the domain H of the m complex variables, which for $m=1$ has to go over into the upper half-plane, (2) the theory of the group Ω of analytic transformations which leave H invariant, (3) the theory of discontinuous subgroups Δ of Ω, culminating in the discussion of a fundamental region for Δ. This geometric and group-theoretical basis of the theory of automorphic functions of m variables is given in the present paper.

The $m=\frac{1}{2}n(n+1)$ complex variables are the elements $z_{kl}=z_{lk}$ $(1 \leqq k \leqq l \leqq n)$ of a symmetric matrix $\mathfrak{Z}=\mathfrak{X}+i\mathfrak{Y}$, where $\mathfrak{X}=\frac{1}{2}(\mathfrak{Z}+\bar{\mathfrak{Z}})$ and $\mathfrak{Y}=(1/2i)(\mathfrak{Z}-\bar{\mathfrak{Z}})$ are the real and imaginary parts of $\mathfrak{Z}$. Inequalities, needed to define certain areas, are introduced by the definition: $\mathfrak{H}>0$ means that $\mathfrak{H}=\bar{\mathfrak{H}}$ is the matrix of a positive definite Hermitian form. If $\mathfrak{E}$ is the unit matrix of n rows and columns, then the domain E with $\mathfrak{E}-\mathfrak{Z}\bar{\mathfrak{Z}}>0$ for symmetric $\mathfrak{Z}$ corresponds to the unit circle, and the domain H with $\mathfrak{Y}=(1/2i)(\mathfrak{Z}-\bar{\mathfrak{Z}})>0$ to the upper half-plane in the case $m=n=1$. The domain H is mapped onto itself by all substitutions

$$\mathfrak{W}=(\mathfrak{A}\mathfrak{Z}+\mathfrak{B})(\mathfrak{C}\mathfrak{Z}+\mathfrak{D})^{-1}$$

of the (nonhomogeneous) symplectic group Ω [cf. H. Weyl, The Classical Groups, Princeton University Press, Princeton, N. J., 1939; these Rev. 1, 42], where $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ are real matrices of n rows and columns satisfying

$$\begin{pmatrix} \mathfrak{A}' & \mathfrak{C}' \\ \mathfrak{B}' & \mathfrak{D}' \end{pmatrix}\begin{pmatrix} 0 & \mathfrak{E} \\ -\mathfrak{E} & 0 \end{pmatrix}\begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{pmatrix}=\begin{pmatrix} 0 & \mathfrak{E} \\ -\mathfrak{E} & 0 \end{pmatrix}.$$

The matrix

$$\mathfrak{M}=\begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{pmatrix}$$

of $2n$ rows and columns is called a symplectic matrix. Theorem 1 states that every analytic mapping of H onto itself is symplectic. Since the transformation

$$\mathfrak{Z}_0=(\mathfrak{Z}-i\mathfrak{E})(\mathfrak{Z}+i\mathfrak{E})^{-1}$$

established a one-to-one correspondence between the points $\mathfrak{Z}$ of H and $\mathfrak{Z}_0$ of E, it suffices to prove that an analytic mapping $\mathfrak{Z}_0 \to \mathfrak{W}_0$ of E into itself with the fixed point 0 is always represented by

$$\mathfrak{W}_0=\mathfrak{U}'\mathfrak{Z}_0\mathfrak{U}$$

with constant unitary $\mathfrak{U}$. This is established by the introduction of an auxiliary complex variable t in the unit circle $|t| \leqq 1$.

The further discussion is based on a generalization of Poincaré space. For two points $\mathfrak{Z}, \mathfrak{Z}_1$ of H the matrix

$$\mathfrak{R}(\mathfrak{Z}, \mathfrak{Z}_1)=(\mathfrak{Z}-\mathfrak{Z}_1)(\mathfrak{Z}-\bar{\mathfrak{Z}}_1)^{-1}(\bar{\mathfrak{Z}}-\bar{\mathfrak{Z}}_1)(\bar{\mathfrak{Z}}-\mathfrak{Z}_1)^{-1}$$

generalizes a cross ratio. The characteristic roots of $\mathfrak{R}$ are real numbers of the interval $0 \leqq r<1$. Two pairs of points $\mathfrak{Z}, \mathfrak{Z}_1$ and $\mathfrak{W}, \mathfrak{W}_1$ of H can be mapped on each other if and only if $\mathfrak{R}(\mathfrak{Z}, \mathfrak{Z}_1)$ and $\mathfrak{R}(\mathfrak{W}, \mathfrak{W}_1)$ have the same characteristic roots. Any pair $\mathfrak{Z}, \mathfrak{Z}_1$ can be transformed into $i\mathfrak{E}$, $i\mathfrak{T}$, $\mathfrak{T}$ being a diagonal matrix with real diagonal elements $t_1, \cdots, t_n$, $1 \leqq t_1 \leqq \cdots \leqq t_n$. The Hermitian differential form

$$ds^2=\text{trace}\,(\mathfrak{Y}^{-1}d\mathfrak{Z}\mathfrak{Y}^{-1}d\bar{\mathfrak{Z}})$$

is invariant under Ω and defines a Riemann metric in H. Any two points of H can be connected by one and only one geodesic in this metric. The distance $\rho(\mathfrak{Z}, \mathfrak{Z}_1)$ of $\mathfrak{Z}$ and $\mathfrak{Z}_1$ is given by

$$\rho^2=\text{trace}\left(\log\frac{1+\mathfrak{R}^{\frac{1}{2}}}{1-\mathfrak{R}^{\frac{1}{2}}}\right)^2, \qquad \mathfrak{R}=\mathfrak{R}(\mathfrak{Z}, \mathfrak{Z}_1).$$

A generalization of the Gauss-Bonnet formula by Allendoerfer and Fenchel yields the Euler characteristic χ of a

closed manifold F in the above Riemann metric:

$$\chi = c_n(-\pi)^{-\ln(n+1)}\int_F dv,$$

where dv is the Euclidean volume element in the space of $\mathfrak{X}, \mathfrak{Y}^{-1}$ and c_n a certain rational number depending only on n. Let Δ be a discontinuous subgroup of Ω and let $D_1, D_2, \cdots$ be its elements, D_1 being the identity. Let $\mathfrak{Z}_k$ be the image $D_k(\mathfrak{Z}_1)$ of $\mathfrak{Z}_1$, $\mathfrak{Z}_1$ being taken as a point of H which is not a fixed point under any transformation D_k $(k=2, 3, \cdots)$. A fundamental region F for Δ can then be defined as consisting of all those points $\mathfrak{Z}$ of H for which

$$\rho(\mathfrak{Z}, \mathfrak{Z}_1) \leqq \rho(\mathfrak{Z}, \mathfrak{Z}_k)$$

for all $k=2, 3, \cdots$. This F is a "star" with respect to $\mathfrak{Z}_1$, that is, if $\mathfrak{Z}$ belongs to F then the whole geodesic arc $\mathfrak{Z}\mathfrak{Z}_1$ also belongs to F.

In analogy to Fuchsian groups a discontinuous symplectic group Δ is said to be of the first kind if there exists a fundamental domain F such that (1) every compact domain in H is covered by a finite number of images of F, (2) only a finite number of images of F are neighbors of F, (3) the integral $V(\Delta)=\int_F dv$ converges. The group Δ is certainly of the first kind if H is compact relative to Δ, that is, if there exists for every infinite sequence of points $\mathfrak{Z}^{(k)}$ $(k=1, 2, \cdots)$ in H at least one compact sequence of images $\mathfrak{Z}_{l_k}{}^{(k)}$ under Δ.

The second and longer part of the paper is devoted to the discussion of special discontinuous symplectic groups defined by arithmetical properties. Let K be a totally real algebraic field of finite degree h, let r be a totally positive number of K; then the field $K_0=K(\sqrt{-r})$ is totally imaginary and of degree $2h$. Let $\mathfrak{H}$ be a Hermitian and $\mathfrak{G}$ be a nonsingular symmetric matrix, both of order $2n$, and with elements of K_0 and let all conjugates of $\mathfrak{H}$, except $\mathfrak{H}$ and $\overline{\mathfrak{H}}$, be positive. Let us further assume the relationship

$$\mathfrak{H}\overline{\mathfrak{G}}^{-1}\overline{\mathfrak{H}} = s\mathfrak{G}$$

with a positive s of K. The numbers r and s can be taken as integers. Then the matrices $\mathfrak{U}$ with integral elements of K_0 satisfying the conditions

$$\mathfrak{U}'\mathfrak{G}\mathfrak{U}=\mathfrak{G}, \quad \mathfrak{U}'\mathfrak{H}\overline{\mathfrak{U}}=\mathfrak{H}$$

form a multiplicative group $\Lambda=\Lambda(\mathfrak{G}, \mathfrak{H})$ which is discontinuous and discrete. There exists a constant matrix $\mathfrak{C}$ (not unique) such that the matrices $\mathfrak{M}=\mathfrak{C}^{-1}\mathfrak{U}\mathfrak{C}$ are symplectic. Identifying $\mathfrak{M}$ and $-\mathfrak{M}$, we have a discontinuous subgroup $\Delta(\mathfrak{G}, \mathfrak{H})$ of Ω. The most important case is the modular group Γ obtained by

$$\mathfrak{G}=\begin{pmatrix} 0 & \mathfrak{E} \\ -\mathfrak{E} & 0 \end{pmatrix}, \quad \mathfrak{H}=i\mathfrak{G}, \qquad h=1; r=1.$$

The group $\Delta(\mathfrak{G}, \mathfrak{H})$ is of the first kind. Instead of using the above defined fundamental region, which may have a very complicated shape, the author introduces fundamental regions based on special arithmetic properties of $\Delta(\mathfrak{G}, \mathfrak{H})$, using for the modular group Minkowski's theory of reduction of quadratic forms and for general $\Delta(\mathfrak{G}, \mathfrak{H})$ P. Humbert's extension of Minkowski's theory to an arbitrary field K_0. For this purpose the space H is first mapped on the space S of all symplectic positive symmetric matrices $\mathfrak{S}$.

The volume of the fundamental region of the modular group is

$$(*) \qquad V_n = 2\prod_{k=1}^{n}\{(k-1)!\pi^{-k}\zeta(2k)\}.$$

The proof is obtained by recursion from n to $n-1$; Riemann's ζ-function comes into play by a summation over lattice points. The formula (*) admits of an interesting interpretation in terms of densities of real and of p-adic matrix solutions of certain conditions. For the commensurability of two groups $\Delta(\mathfrak{G}, \mathfrak{H})$ and $\Delta(\mathfrak{G}_1, \mathfrak{H}_1)$, that is, for the existence of subgroups of finite indices which are conjugate in Ω, a necessary and sufficient condition is given in theorem 13. For $n=2$ another example of a discontinuous symplectic group $\Delta(\mathfrak{T})$ of the first kind is furnished by the theory of units of quinary quadratic forms of signature 2, 3.

Even a lengthy review can indicate only a few outstanding marks among the abundant results and methods of this important paper. *H. Rademacher* (Swarthmore, Pa.).

Siegel, Carl Ludwig. Note on automorphic functions of several variables. Ann. of Math. (2) **43**, 613–616 (1942). [MF 7393]

The notations are the same as in the preceding review. Let Δ be a discontinuous group of linear transformations which map E, the "unit circle" $\mathfrak{E}-\mathfrak{Z}\overline{\mathfrak{Z}}>0$, onto itself. Assume moreover that E is compact relative to Δ. Then there exists a fundamental region of finite diameter δ, the definition of distance being taken from the preceding paper. An analytic function $f(\mathfrak{Z})$ of $\frac{1}{2}n(n+1)$ independent complex variables z_{kl} $(1\leqq k\leqq l\leqq n)$ is called an automorphic form with the group Δ if it is regular in E and satisfies there the transformation relations

$$f(\mathfrak{Z}^*) = v(\mathfrak{A}, \mathfrak{B})\cdot|\overline{\mathfrak{B}}\mathfrak{Z}+\overline{\mathfrak{A}}|^{-g}f(\mathfrak{Z})$$

for any transformation

$$\mathfrak{Z}^* = (\mathfrak{A}\mathfrak{Z}+\mathfrak{B})(\overline{\mathfrak{B}}\mathfrak{Z}+\overline{\mathfrak{A}})^{-1}$$

belonging to Δ. The set $L(\Delta, g, v)$ of all such functions $f(\mathfrak{Z})$ of given weight g and given multiplier system v forms a vector space. The paper gives an upper estimate of the dimension of $L(\Delta, g, v)$, depending only on n and δ. The proof of the essential lemma is based on an application of the principle of the maximum on an analytic function of a single complex variable z in a circle $|z|^2\leqq t$. The paper closes with a remark about the algebraic field of automorphic functions belonging to Δ. *H. Rademacher.*

Bergman, Stefan and Spencer, D. C. A property of pseudo-conformal transformations in the neighborhood of boundary points. Duke Math. J. **9**, 757–762 (1942). [MF 7926]

Die Verfasserbeweisen den folgenden Satz: $\mathfrak{P}^2$ sei ein in $E[r_k=|z_k|>0, k=1, 2]$ liegender beschränkter Bereich mit $\int\int_{\mathfrak{P}} dr_1 dr_2/r_1r_2=\infty$; ferner sei $\mathfrak{p}$ der (einzige) Randpunkt, für den entweder $r_1=0$ oder $r_2=0$. (Der obere Index bezeichnet die Dimension; $E[\ ]$ gibt die Punktmenge an, welche der in der Klammer stehenden Bedingung genügt). Jedem (r_1, r_2) aus $\mathfrak{P}^2$ sei ein gewisser Teilbereich $\mathfrak{S}_1{}^2(r_1, r_2)$ von $E[z_k=r_ke^{i\varphi_k}]$ zugeordnet, der von einer geschlossenen Jordankurve $\mathfrak{C}_1{}^1(r_1, r_2)$ begrenzt werde; $\mathfrak{R}^4$ sei die Summe aller $\mathfrak{S}_1{}^2(r_1, r_2)$. Jedes $\mathfrak{C}_1{}^1(r_1, r_2)$ werde ferner willkürlich in vier Teile $\mathfrak{C}_{1\gamma}{}^1(r_1, r_2)$, $\gamma=1, \cdots, 4$, zerlegt. Die $\mathfrak{C}_{1\gamma}{}^1(r_1, r_2)$ mögen durch eine in $\mathfrak{R}^4$ reguläre, pseudokonforme Abbildung (eineindeutig und stetig) auf die $\mathfrak{C}_{2\gamma}{}^1(r_1, r_2)$ abgebildet werden. Bezeichnet man dann noch mit $\mathfrak{E}_\gamma$ die Menge aller Grenzpunkte der $\mathfrak{C}_{2\gamma}(r_1, r_2)$ für $(r_1, r_2)\to\mathfrak{p}$ (auf jedem beliebigen Wege innerhalb $\mathfrak{P}^2$), so gilt unter gewissen Voraussetzungen über die beiden Abbildungsfunktionen $w_k=f_k(z_1, z_2)$, dass entweder der Durchschnitt von $\mathfrak{E}_1$ und $\mathfrak{E}_3$ oder der Durchschnitt von $\mathfrak{E}_2$ und $\mathfrak{E}_4$ nicht leer ist.

P. Thullen (Quito).

Fourier Series and Generalizations, Integral Transforms

Loo, Ching Tsün. The absolute summability of power series and Fourier series. Duke Math. J. **10**, 271–276 (1943). [MF 8467]

The purpose of the paper is to prove the following theorems. (I) Let $f(z)=\sum_0^\infty c_n z^n$ be regular for $|z|<1$. If $f(z)$ satisfies the condition

$$(1)\quad \left((2\pi)^{-1}\int_0^{2\pi}|f'(re^{i\theta})|^p d\theta\right)^{1/p} = O((1-r)^{-1+1/p}(\log(1-r)^{-1})^{-q}),\quad q>1,$$

and if $1<p\leqq 2$, then $\sum|c_n|<\infty$. (II) If (1) is satisfied and $p>2$ the series $\sum c_n e^{n\theta}$ is summable $|C,\alpha|$ for every $\alpha>\frac{1}{2}-1/p$. (III) Let

$$(2)\quad u(\theta)\sim\sum a_n\cos n\theta+b_n\sin n\theta.$$

If $u(A)$ satisfies the condition

$$(3)\quad \left(\int_0^{2\pi}|u(\theta+h)-u(\theta)|^p d\theta\right)^{1/p} = O(h^{1/p}(\log h^{-1})^{-q}),\quad q>1,$$

and if $1<p\leqq 2$, the series (2) is absolutely convergent. (IV) If (3) is satisfied and $p>2$, the series (2) is summable $|C,\alpha|$ for every $\alpha>\frac{1}{2}-1/p$. Theorem (III) has also been proved by M.-T. Cheng [Duke Math. J. **9**, 803–810 (1942); these Rev. **4**, 156]. *R. Salem* (Cambridge, Mass.).

Chow, Hung Ching. A further note on a theorem of O. Szász. J. London Math. Soc. **17**, 177–180 (1942). [MF 8268]

O. Szász [Duke Math. J. **4**, 401–407 (1938)] and G. Maruyama [Tôhoku Math. J. **46**, 68–74 (1939); these Rev. **1**, 138] have established formulae connecting the jump of a function $f(x)$ with the Fejér sum of the series conjugate to the Fourier series of $f(x)$. The author has extended these formulae to the Cesàro sums of any order $\alpha>0$ of the conjugate series [J. London Math. Soc. **16**, 23–27 (1941); these Rev. **3**, 105]. Here he gives a new and shorter proof of his results. *R. Salem* (Cambridge, Mass.).

Szász, Otto. On the partial sums of Fourier series at points of discontinuity. Trans. Amer. Math. Soc. **53**, 440–453 (1943). [MF 8418]

The first part of the paper deals with the properties of transforms of the type

$$T_n=\sum_{\nu=1}^{n}\rho_n^{\nu}b_\nu\sin\nu\theta_n,$$

where $\rho_n\to 1$ and $\theta_n\to 0$. Necessary and sufficient conditions are given for the existence of the limit of T_n in the following cases: (i) nb_n tends to a limit; (ii) the sequence $\{nb_n\}$ is summable $(C,1)$; (iii) the sequence $\{nb_n\}$ is summable (C,K) $(K>1)$. In the second part of the paper the results are applied to Fourier series to show generalized Gibbs phenomena of the following type. Let

$$f(\theta)\sim\sum a_n\cos n\theta+b_n\sin n\theta$$

and let $s_n(\theta)$ be the nth partial sum; if $\sum_1^n\nu b_\nu>-pn$ (for some $p>0$ and for all n) and if j is the generalized jump (Szász) of $f(\theta)$ at $\theta=0$, then

$$s_n(\theta_n)\to(j/\pi)\int_0^{\lambda\pi}t^{-1}\sin t dt,$$

where $n\theta_n-\lambda\pi=O(n^{-1})$, λ being a positive integer.
R. Salem (Cambridge, Mass.).

Szegö, G. On the oscillation of differential transforms. IV. Jacobi polynomials. Trans. Amer. Math. Soc. **53**, 463–468 (1943). [MF 8420]

Let α, β, c be real numbers and let

$$\vartheta\equiv(1-x^2)D^2+[\beta-\alpha-(\alpha+\beta+2)x]D,\quad D=d/dx.$$

Hille has shown [same Trans. **52**, 463–497 (1942); these Rev. **4**, 97] that, if α, β, $c\geqq 0$, then (i) the differential operator $\vartheta-c$ does not diminish the number of sign changes in the interval $-1<x<+1$; (ii) if $f(x)$ has derivatives of all orders in the interval $-1\leqq x\leqq+1$, and if the number of the sign changes of the functions $(\vartheta-c)^k f(x)$ remains not greater than N for $k=1, 2, \cdots$, then $f(x)$ is a polynomial of degree not greater than N. Szegö now shows that (i) holds for $\alpha>-1$, $\beta>-1$, $c\geqq 0$ and that if in (ii) the assumption $\alpha, \beta\geqq 0$ is dropped then, under the remaining hypotheses, $f(x)$ is a polynomial of degree $N+\gamma$, where $\gamma=\gamma(\alpha, \beta, c)$.
A. Zygmund (South Hadley, Mass.).

Wintner, Aurel. On the Ω-matrices of Toeplitz. Amer. J. Math. **64**, 669–676 (1942). [MF 7170]

The theory of Ω-matrices was developed by Toeplitz [Math. Ann. **70**, 351–376 (1911)] for the case of analytic Fourier series. The restriction of analyticity is essential for the method of Toeplitz. The object of the present note is to develop the theory in full generality. If f is a measurable function of absolute value one, $\Omega(f)$ denotes the matrix $\|w_{nm}\|$, where w_{nm} is the mth Fourier coefficient of f^n. The author discusses, in particular, conditions for the boundedness and nonsingularity of Ω. As an application, he shows that, if f gives a continuous one-to-one transformation of the unit circle in itself, its Ω matrix will be nonsingular if and only if f is of bounded "distortion," that is, if the Lipschitz conditions

$$A|x_1-x_2|\leqq|f(x_1)-f(x_2)|\leqq A'|x_1-x_2|$$

are satisfied. *F. Bohnenblust* (Princeton, N. J.).

Hellinger, E. D. and Wall, H. S. Contributions to the analytic theory of continued fractions and infinite matrices. Ann. of Math. (2) **44**, 103–127 (1943). [MF 8079]

The main object of this paper is to extend to the "J-continued fraction"

$$(*)\quad \frac{1|}{|b_1 z}-\frac{a_1^2|}{|b_2 z}-\frac{a_2^2|}{|b_3 z}-\cdots,$$

$a_p>0$ real, b_p complex, $\Im(b_p)\geqq 0$, $\Im(z)>0$, results obtained earlier by various investigators for the case of b_p real. The authors introduce the notions of complete and simple convergence of (*) and establish sufficient conditions for the former. They study also the nature of the function $f(z)$ represented by (*), its asymptotic and integral representation, more generally, the integral representation

$$f(z)=\int_{-\infty}^{\infty}d\psi(u)/(z+u),$$

$\psi(u)\uparrow$, $\psi(+\infty)-\psi(-\infty)=1$, analogous to the classical representation of Stieltjes. Finally, the relation is indicated

of the foregoing to the "moment problem": $\int_{-\infty}^{\infty} u^p d\psi(u) = c_p$, real, $p = 0, 1, 2, \cdots$. The authors employ infinite sequences of linear transformations in one variable, also linear transformations in infinitely many variables and infinite J-matrices arising from (*):

$$J = \begin{bmatrix} b_1+z & -a_1 & 0 & 0 & \cdots \\ -a_1 & b_2+z & -a_2 & 0 & \cdots \\ 0 & -a_2 & b_3+z & -a_3 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

J. A. Shohat (Philadelphia, Pa.).

Differential Equations

Curry, H. B. The Heaviside operational calculus. Amer. Math. Monthly **50**, 365–379 (1943). [MF 8488]

A simple introduction to the elements of the operational calculus avoiding integral transforms. The subject is treated in a purely algebraic way and is exceedingly simple and elementary. As the author states, "this advantage, of course, implies a restriction on the scope of the treatment, because it is limited to the rational aspects such as arise from ordinary linear differential equations with constant coefficients. For the more general cases of partial differential equations, fractional operators, etc., the theory of integral transforms is doubtless unavoidable." *W. Feller.*

Bulgakov, B. V. Periodic processes in free pseudo-linear oscillatory systems. J. Franklin Inst. **235**, 591–616 (1943). [MF 8480]

This is a translation into English of a paper [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **6**, 263–280 (1942); these Rev. **4**, 142] by the same author.

N. Levinson (Cambridge, Mass.).

Bulgakov, B. V. On the application of the method of van der Pol to pseudo-linear oscillatory systems with many degrees of freedom. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **6**, 395–410 (1942). (Russian. English summary) [MF 8360]

The author treats the system

$$(*) \qquad \sum_{k=1}^{n} f_{jk}(D, \mu) x_k = \mu \phi_j(x_1, \cdots, x_n), \quad j = 1, 2, \cdots, n,$$

where $D = d/dt$, μ is small and $f_{jk}(D, \mu)$ are polynomials in D. Approximate solutions of (*) are obtained in quite explicit form. Application is made to the study of the effect of friction in the gimbals on the precession and nutation of a gyro-pendulum. *N. Levinson* (Cambridge, Mass.).

Quade, W. Ein neues Verfahren der schrittweisen Näherungen zur Lösung von $y' = f(x, y)$. Math. Z. **48**, 324–368 (1942). [MF 8557]

The differential equation (1) $y' = f(x, y)$ is treated under the assumptions that $f(x, y)$ is continuous in x and satisfies a Lipschitz condition in y in a convex region B (the Lipschitz constant being denoted by L). For Picard's method of successive approximations a new method of finding the interval of convergence is given. Next, the paper shows how to construct a monotone sequence of over- (under-) functions whose limiting value is a solution of (1). The method of constructing the monotone sequence is formally simple. Thus, if $u_n(x)$ is an over- (under-) function, then $u_{n+1}(x)$ is given by

$$u_{n+1} = u_n - \int_{x_0}^{x} e^{-L(x-t)}[u_n'(t) - f(t, u_n(t))]dt.$$

F. G. Dressel (Durham, N. C.).

***Bleuler, Konrad. Über den Rolle'schen Satz für den Operator $\Delta u + \lambda u$ und die damit zusammenhängenden Eigenschaften der Green'schen Funktion.** Thesis Eidg. Techn. Hochsch. Zürich, 1942. 37 pp. [MF 8380]

Let u be a function defined in a region B, and let it satisfy a linear homogeneous boundary condition $R(u) = 0$ on Γ, a set of points, or closed curves, etc., according as B has dimension 1, 2, etc. $D(u)$ is a linear homogeneous operator on u and $B' = B - \Gamma$. The author considers the following theorem: Under suitable assumptions of continuity, if $u = 0$ at a point of B', then $D(u) = 0$ at some point of B'. This theorem has been proved for a number of cases for $R(u)$ and $D(u)$ by Polya. In the present paper, it is shown that it holds for the operator $\Delta u + \lambda u$, and the boundary condition $u = 0$, provided λ is less than λ_1, the first characteristic value of the equation $\Delta u + \lambda u = 0$; if $\lambda > \lambda_1$, the theorem does not hold. It is first shown that the theorem stated above is equivalent to the fact that a Green's function corresponding to the equation $D(u) = 0$ always has a fixed sign. It is then shown that, for $D(u) = \Delta u + \lambda u$, the Green's function has a constant sign for $\lambda < \lambda_1$, but a sign which changes for $\lambda > \lambda_1$. Toward these ends a study is made of the variation of the Green's functions and of the characteristic functions of $\Delta u + \lambda u = 0$, when the boundary is altered. *J. W. Green.*

Biben, Georges. Sur une généralisation d'un théorème de Schwarz. C. R. Acad. Sci. Paris **214**, 603–606 (1942). [MF 7894]

D is a domain in n-dimensional space and P a positive function defined in it. $u_1, u_2, u_3, \cdots$ is a sequence of functions vanishing on the boundary of D and satisfying $\Delta_2 u_n + P u_{n-1} = 0$, $n = 1, 2, \cdots$; $u_0 = 1$; W_n is defined to be $\int_D P \cdot u_n dT$. Using Green's formula the author proves that W_n can be expressed in the form

$$W_n = \int_D P \cdot u_k u_{n-k} dT = \int_D \Delta_1(u_{k+1}, u_{n-k}) dT, \qquad k \leqq n.$$

It is also proved from the positiveness of the Dirichlet integral, $\int_D \Delta_1(\alpha u_n + \beta u_{n+1}) dT$, for all α and β different from zero, that the ratio W_{n+1}/W_n is an increasing function of n.

J. W. Green (Aberdeen, Md.).

Stepanoff, W. Sur l'équation de Laplace et certains systèmes triples orthogonaux. Rec. Math. [Mat. Sbornik] N.S. **11**(53), 204–238 (1942). (French. Russian summary) [MF 8368]

The author makes a systematic derivation of all systems of curvilinear coordinates ρ_1, ρ_2, ρ_3 in three dimensions for which Laplace's equation admits a family of solutions of the form $U_1(\rho_1) \cdot U_2(\rho_2) \cdot U_3(\rho_3)$, where U_i depends on two parameters. The method consists in assuming a metric and examining the conditions imposed on the coefficients of the metric by the fact that Laplace's equation admits solutions of the type stated. Imposition of the condition that the tensor of total curvature vanishes reduces the metric to one of several types. Each type is identified geometrically and the associated classical functions, trigonometric, exponential, Bessel, Legendre, Lamé, Mathieu, etc., obtained. A

similar analysis is made for cylindrical coordinates in four dimensions. *J. W. Green* (Aberdeen, Md.).

Transue, W. R. Representation of subharmonic functions in the neighborhood of a point. Amer. J. Math. 65, 335–340 (1943). [MF 8207]

If D is a domain and u a function subharmonic in D, it is known that it is not always possible to represent u in the form

$$(1) \qquad u(P)=\int_D \log PQ\,d\mu(e_Q)+h(P),$$

where $\mu(e)$ is a positive mass distribution and h is harmonic. It is possible so to represent u in any domain D' contained with its boundary in D. It is also possible to represent u in the form (1) if $\log PQ$ is replaced by $G(P, Q)$, the Green's function for D, in the case u has a harmonic majorant. In the present paper the author considers a $u(P)$ subharmonic in the whole plane except at ∞, and the possibility of representing it in the form

$$(2) \qquad u(P)=\int_D E(P, Q)\,d\mu(e_Q)+h(P)$$

for a properly chosen function E. For E is taken the function

$$E_p(P, Q)=\log PQ-\log OQ+R\left[\frac{P}{Q}+\frac{1}{2}\left(\frac{P}{Q}\right)^2+\cdots+\frac{1}{p}\left(\frac{P}{Q}\right)^p\right].$$

In an annulus ${}_bC_R$ of inner and outer radii b and R, respectively, $u(P)$ can be represented in the form (1). From this it follows that

$$u(P)=\int_{{}_bC_R} E_p\,d\mu+\text{a harmonic function}.$$

It is then proved that, if

$$\lim_{R\to\infty}\int_{{}_bC_R}\frac{d\mu(e_Q)}{|OQ|^{p+1}}$$

exists, then

$$\int_{{}_bC_\infty} E_p(P, Q)\,d\mu(e_Q)$$

exists also, and that u can be expressed in the form (2), where D is ${}_bC_\infty$. By making an inversion, the results can be interpreted with respect to a function subharmonic near a point. *J. W. Green* (Aberdeen, Md.).

Privaloff, I. Quelques applications de l'opérateur généralisé de Laplace. Rec. Math. [Mat. Sbornik] N.S. 11(53), 149–154 (1942). (Russian. French summary) [MF 8366]

The author deduces two important results on subharmonic functions and their generalized Laplace operators. In p dimensional space ($p>2$) he defines the operator by

$$\underline{\Delta}u(Q_0)=\lim_{h\to 0}\left[\frac{1}{w}\int u(P)\,dw-u(Q_0)\right]\cdot\frac{\Gamma(p/2)(p+2)}{\pi^{p/2}(p-2)\cdot h^2},$$

where the integration is taken over a sphere of center Q_0, radius h and volume w. The main result is the following. If (i) u and v are subharmonic in a domain G and $\underline{\Delta}u(Q)<\infty$, $\underline{\Delta}v(Q)<\infty$ for all $Q\in G$, and (ii) $\underline{\Delta}u(Q)=\underline{\Delta}v(Q)$ for almost all $Q\in G$, then $u(Q)=v(Q)+h(Q)$, where $h(Q)$ is harmonic in G. Another theorem shows that, if E is a closed and bounded set of measure zero, and u is subharmonic and bounded from above in a neighborhood of E and upper semicontinuous at the points of E, then $u(Q)$ will be subharmonic in a domain containing E if $\overline{\Delta}u(Q)>-\infty$ everywhere in E except at points of a set of zero capacity. These results yield obviously important corollaries for harmonic functions. The proofs, based on some modern results of the theory of functions of real variables, are straightforward.

František Wolf (Berkeley, Calif.).

Minakshisundaram, S. On expansion in eigenfunctions of boundary value problems. III. The summability problem. J. Indian Math. Soc. (N.S.) 6, 153–167 (1942). [MF 8375]

This paper is a sequel to two papers by the same author [Proc. Indian Acad. Sci., Sect. A. 12, 462–465 (1940); J. Indian Math. Soc. (N.S.) 5, 103–108 (1941); these Rev. 3, 128, 244]. In the (x, y) plane let D be a finite open domain, simply or multiply connected, whose boundary C consists of a finite number of regular curves, and let $\omega_n(x, y)$ be the complete normal eigenfunctions with the eigenvalues $\mu_n>0$, $n=1, 2, 3, \cdots$, corresponding to the boundary value problem

$$\nabla^2\omega+\mu\omega=0 \quad \text{in } D, \qquad \omega(x, y)=0 \quad \text{on } C.$$

Given $f(x, y)$ any measurable function integrable L in D, let

$$a_n=\int\int_D f(x, y)\omega_n(x, y)\,dx\,dy.$$

Given also $\rho>0$ and the circle C_r in D with center at (x, y) and radius $r>0$, let

$$(\xi-x)^2+(\eta-y)^2=t^2\leqq r^2,$$

and let

$$f_p(r; x, y)=f_p(r)=(p/\pi r^2)\int\int_{C_r}(1-t^2/r^2)^{p-1}f(\xi, \eta)\,d\xi\,d\eta.$$

The following are the principal theorems given. (A) If $f(x, y)\in L^2$ in D and if $f_p(r)\to s$ as $r\to 0$, then the series $\sum_{n=1}^{\infty}a_n\omega_n(x, y)$ is summable (R, μ, k) to the sum s for $k>\max(\frac{1}{2}, p+\frac{1}{2})$. (B) If $f(x, y)\in L^2$ in D and if for a point (x, y) in D the series $\sum_{n=1}^{\infty}a_n\omega_n(x, y)$ is summable (R, μ, k), $k\geqq 0$, to the sum s, then $\lim_{r\to 0}f_p(r)=s$ for $p>k+\frac{1}{2}$. (C) If $\sum_{n=1}^{\infty}\mu_n^{\nu}a_n^2$ is convergent for $\nu>\frac{1}{2}$, then $\sum_{n=1}^{\infty}a_n\omega_n(x, y)$ is absolutely convergent at every point of D, and uniformly in every domain lying wholly in D. (D) If $\sum_{n=1}^{\infty}\mu_n a_n^2$ converges, then $\sum_{n=1}^{\infty}a_n\omega_n(x, y)$ converges at a point (x, y) in D if and only if $\lim_{r\to 0}f_1(r)$ exists. When this limit exists it is the value of $\sum_{n=1}^{\infty}a_n\omega_n(x, y)$. *F. W. Perkins.*

Faber, Karl. Über den Zusammenhang der drei Typen von partiellen Differentialgleichungen zweiter Ordnung in zwei Veränderlichen mit gewissen Funktionentheorien. Deutsche Math. 6, 323–341 (1942). [MF 8608]

Associate with the partial differential equation

$$(1) \qquad \frac{\partial^2 F}{\partial x^2}-s^2\frac{\partial^2 F}{\partial y^2}=0$$

the "complex" variable $z=x+sy$, and call $f(z)=q(x, y)+sp(x, y)$ "analytic" if $q_x=p_y$, $q_y=s^2p_x$ (p_x, p_y, q_x, q_y continuous). For the case of ordinary complex variables ($s^2=i^2=-1$) the association between equation (1) and analytic functions is well known. The present paper develops similar associations between "analytic" functions of the

"complex" variable $z=x+sy$ and the equation (1) in the cases $s^2=1$, and $s^2=0$. Some of the results obtained are: each "analytic" function has at least a first order differential coefficient of a characteristic of the equation (1); the line integral of an "analytic" function exists and is itself "analytic"; each solution of (1) leads to an "analytic" function $f(z)=F_y+sF_x$ and conversely. *F. G. Dressel.*

Awberry, J. H. A problem in two-dimensional flow. Proc. Phys. Soc. 55, 202–203 (1943). [MF 8444]

Formal solution of $xu_t=(xu_x)_x$ with $0\leqq x\leqq l$, $u_x(0,t)=0$, $u_x(l,t)=a$ and $u(x,0)=b$, where a and b are given constants. *W. Feller* (Providence, R. I.).

Carslaw, H. S. Note on the paper "The temperature distribution around a spherical hole in an infinite conducting medium," by Messrs. Pugh and Harris in the Phil. Mag. p. 661 (Sept. 1942). Philos. Mag. (7) 34, 288 (1943). [MF 8413]

Further references are given to the solution of this problem [cf. these Rev. 4, 46]. *R. V. Churchill.*

Functional Analysis

Bourgin, D. G. Some properties of Banach spaces. Amer. J. Math. 64, 597–612 (1942). [MF 6952]

The paper is concerned with some of the interrelations between such concepts as weak compactness, weak completeness, weak convergence and the property of Helly. In particular, the author considers these concepts as of order $\aleph_\alpha$; for example, a set E has the extended Helly property of order $\aleph_\alpha$ in case

$$|\sum_1^N \lambda_{\rho i}C_{\rho i}|\leqq M\|\sum_1^N \lambda_{\rho i}f^{\rho i}\|$$

for arbitrary sets $(\lambda_{\rho i}|i=1,\cdots,N)$ of real numbers implies that the system of at most $\aleph_\alpha$ equations $f^\rho(x)=c_\rho$ has a solution x_ϵ with $\|x_\epsilon\|\leqq M+\epsilon$ for $\epsilon>0$. It is shown that $w(E)$ compactness of order $\aleph_\alpha$ implies the extended Helly property of order $\aleph_\alpha$, and that, if the extended Helly property holds for all orders, then the space is $w(E)$ complete. As a consequence of this latter result E is reflexive if and only if it has the extended Helly property. A result of considerable interest is the following: If f^ρ is total, x^ρ is a directed set of order not greater than $\aleph_\alpha$ in a $w(E)$ compact set of order $\aleph_\alpha$ and $f^\rho(x^\rho)$ converges to $f^\rho(x_0)$ for each μ, then x^ρ is $w(E)$ convergent to x_0. *H. H. Goldstine* (Philadelphia, Pa.).

Bohnenblust, Frederic. A characterization of complex Hilbert spaces. Portugaliae Math. 3, 103–109 (1942). [MF 7083]

It is shown that a complex Banach space is a Hilbert space (perhaps not separable) if for every two dimensional subspace there exists a projection of norm one of the entire space on the subspace. For real spaces this result was proved by Kakutani [Jap. J. Math. 16, 93–97 (1939); these Rev. 1, 146] and Phillips [Bull. Amer. Math. Soc. 46, 930–933 (1940); these Rev. 2, 220]. *N. Dunford.*

Schoenberg, I. J. On local convexity in Hilbert space. Bull. Amer. Math. Soc. 48, 432–436 (1942). [MF 6709]

A set M in a normed linear vector space is said to be locally convex in case for every $p\in M$ there is a neighborhood of p whose intersection with M is convex. Tietze has shown that a closed connected locally convex set in Euclidean space is convex. The present paper gives an elementary proof of this theorem for any normed linear vector space. *N. Dunford* (New Haven, Conn.).

Yosida, Kôsaku. Vector lattices and additive set functions. Proc. Imp. Acad. Tokyo 17, 228–232 (1941). [MF 7137]

An especially short proof of the theorem of Radon-Nikodym is given, and followed by a generalization to an arbitrary countably additive vector lattice L. Relative to any positive element l in L, any element $f\in L$ has an "absolutely continuous" part with a spectral representation and a "singular" remainder. This result is implicit in recent work by other authors (notably Freudenthal, Riesz and Kakutani), but the present treatment is remarkably direct. *G. Birkhoff* (Cambridge, Mass.).

Bochner, S. Completely monotone functions in partially ordered spaces. Duke Math. J. 9, 519–526 (1942). [MF 7334]

Let S be a partially ordered vector space in which the existence of an upper bound for a monotone increasing sequence implies the existence of a least upper bound. The author shows that the Hausdorff-Bernstein-Widder theorem holds in S. Specifically, let t and α be real numbers $(0\leqq t<\infty;\ 0<\alpha<\infty)$, and let S^* consist of the positive elements of S and zero. Then the transformation

$$T(\alpha)=\int_0^\infty e^{-\alpha t}dE(t)$$

takes all monotonically increasing functions $E(t)$ with range in S^* into all completely monotone functions $T(\alpha)$ with range in S^*. Moreover the transformation is essentially one-to-one. As a special case, the author points out that this theorem implies a recent result of E. Hille [Proc. Nat. Acad. Sci. U.S.A. 24, 159–161 (1938)]. *R. H. Cameron* (Cambridge, Mass.).

Lorch, Edgar R. The spectrum of linear transformations. Trans. Amer. Math. Soc. 52, 238–248 (1942). [MF 7118]

The paper concerns itself with the question of the reducibility of a bounded linear transformation mapping a complex vector space into itself, as a first step toward determining the structure of such a transformation. The principal method used is that of a contour integral of the resolvent of T, the fundamental projection being the integral

$$P=(2\pi i)^{-1}\int_C dz/(zI-T)$$

calculated along a simply closed curve in the resolvent set of T. It is shown, among other things, that, if $Tf=\lambda f$ $(f\neq 0)$, then $Pf=f$ or $Pf=0$ according as λ is inside or outside C; if λ is in the continuous spectrum and f_n is such that $\|f_n\|=1$, $\|Tf_n-\lambda f_n\|\to 0$, then $\|Pf_n-f_n\|\to 0$ if λ is within C and $\|P_nf\|\to 0$ if λ is without C; $P=0(I)$ is solvable if and only if every point interior (exterior) to C is in the resolvent set of T. It is shown that the correspondence between P and the set interior to C is a homomorphism but that even in reflexive spaces the homomorphism cannot be extended for denumerable sums and products. *H. H. Goldstine* (Philadelphia, Pa.).

Smiley, M. F. and Transue, W. R. Applications of transitivities of betweenness in lattice theory. Bull. Amer. Math. Soc. **49**, 280–287 (1943). [MF 8225]

As in an earlier paper [Pitcher and Smiley, Trans. Amer. Math. Soc. **52**, 95–114 (1942); these Rev. **4**, 87], the relation $(a, b, c)K$ is defined in any lattice to mean $(a \cap b) \cup (b \cap c) = b = (a \cup b) \cap (b \cup c)$. Three kinds of abstract systems are characterized in part in terms of this triadic "betweenness" relation: those metric spaces which can be made into metric lattices (this in terms of metric betweenness), modular lattices with betweenness relation, and lattices with betweenness relation. The postulates are not purely abstract: they involve either the element 0 or the lattice operations. *G. Birkhoff* (Cambridge, Mass.).

Smiley, M. F. A comparison of algebraic, metric, and lattice betweenness. Bull. Amer. Math. Soc. **49**, 246–252 (1943). [MF 8219]

The author correlates the lattice betweenness relation discussed in the paper by Smiley and Transue reviewed above, the metric betweenness relation

$$\delta(a, b)+\delta(b, c)=\delta(a, c),$$

and the vector betweenness relation

$$b=\lambda a+(1-\lambda)c \qquad [0\leqq\lambda\leqq 1].$$

He points out that a linear metric space is strictly convex if and only if in it metric and vector betweenness coincide; a Banach lattice is an abstract L-space (in the terminology of the reviewer's "Lattice Theory" [Amer. Math. Soc. Colloquium Publ., v. 25, New York, 1940; these Rev. **1**, 325]) if and only if metric and lattice betweenness coincide; lattice betweenness and vector betweenness coincide only for the real number system. *G. Birkhoff.*

Theory of Probability

Chung, Kai Lai. Generalization of Poincaré's formula in the theory of probability. Ann. Math. Statistics **14**, 63–65 (1943). [MF 8250]

Let $n=n_1+n_2+\cdots+n_r$ events be grouped in r sets: $E_{11}, E_{12}, \cdots, E_{1n_1}; E_{21}, E_{22}, \cdots, E_{2n_2}; \cdots; E_{r1}, E_{r2}, \cdots, E_{rn_r}$, and let $m_1, m_2, \cdots, m_r$ be given integers such that $0\leqq m_i\leqq n_i$ for $i=1, 2, \cdots, r$. The author derives a formula for the probability that out of the first set of n_1 events exactly m_1 events occur, and out of the second set of n_2 events exactly m_2 events occur, and so on. The problem obtained from the foregoing by replacing the words "exactly" by the words "at least" is also solved, and some other modifications are indicated. *Z. W. Birnbaum.*

Nair, A. N. Krishnan. On the probability of obtaining k sets of consecutive successes in n trials. Math. Student **10**, 83–84 (1942). [MF 7957]

Borel, Émile. Sur l'emploi du théorème de Bernoulli pour faciliter le calcul d'une infinité de coefficients. Application au problème de l'attente à un guichet. C. R. Acad. Sci. Paris **214**, 452–456 (1942). [MF 7857]

The author develops a method for computing certain probabilities and points out that this method can be applied to other problems of a similar nature. The particular problem considered is concerned with a sequence of points A_i distributed along a line in accordance with the Poisson law, the unit of length being so chosen that the mean density of the sequence is one. Consider a set of contiguous intervals such that each is of length σ and the left hand endpoint of the interval on the extreme left is one of the points A_i. Such a set is said to form a series provided the number of points A_i lying in these intervals (including the initial point) is equal to the number of intervals and provided no proper subset of contiguous intervals with the same initial point has this property. The probability p_n of obtaining a series of length n is $a_n\sigma^{n-1}e^{-n\sigma}$, where a_n is independent of σ. The probability p_∞ that a series will not terminate is 0 when $\sigma<1$. When this is the case, σ can be expanded in a power series in $z=\sigma e^{-\sigma}$ with coefficients a_n. Hence these coefficients can be calculated by contour integration. The quantities $p_1, p_2, \cdots, p_\infty$ can then be calculated for all values of σ. *A. H. Copeland* (Ann Arbor, Mich.).

Curtiss, J. H. Convergent sequences of probability distributions. Amer. Math. Monthly **50**, 94–105 (1943). [MF 8052]

The author proves the following convergence test, connected with the central limit theorem. Let X_n be a random variable depending on a parameter n; let μ_n be its mean, σ_n^2 its variance, $F_n(t)$ the distribution function of $(X_n-\mu_n)/\sigma_n$, and $H_n(\alpha)$ the semi-invariant generating function of X_n, that is to say, $H_n(\alpha)=\log E(e^{\alpha X_n})$. If either of the equations

$$\lim_{n\to\infty} H_n''(\alpha/\sigma_n)=1, \quad \lim_{n\to\infty} H_n'''(\alpha/\sigma_n)/\sigma_n^3=0$$

holds uniformly in some neighborhood of $\alpha=0$, then $F_n(t)$ tends to the reduced Gaussian distribution. As applications, the author treats the binomial, the Poisson distribution, the Γ-distribution, the z-distribution and the distribution of linear forms and of positional means. *W. Feller.*

Bhattacharyya, B. C. A note on the tractrix and the cycloid as statistical distribution curves. Bull. Calcutta Math. Soc. **34**, 105–108 (1942). [MF 7575]

The author considers the following two special cases of the problem of constructing stochastic variables with preassigned probability distributions: (1) by combining chi-squares he obtains a variable whose probability distribution is part of a tractrix defined by $dy/dx=-y/(a^2-y^2)^{\frac{1}{2}}$ and (2) using a function of Student's t he defines a variable which has a part of a cycloid as probability distribution. *Z. W. Birnbaum* (Seattle, Wash.).

Radhakrishna Rao, C. On the volume of a prismoid in n-space and some problems in continuous probability. Math. Student **10**, 68–74 (1942). [MF 7954]

This paper offers a method for computing volumes bounded by hyperplanes in n-space. The method is then applied to the solution of problems in geometrical probability. In each case the desired probability is given as a ratio of two such n-dimensional volumes. *A. H. Copeland* (Ann Arbor, Mich.).

Chandrasekhar, S. Stochastic problems in physics and astronomy. Rev. Modern Phys. **15**, 1–89 (1943). [MF 8303]

The author has written a valuable detailed survey of problems in physics centering around random movements of particles. A bibliography is included. The survey starts with a discussion of random walk problems, leading to Brownian motion of particles. The theory of Brownian motion is treated both in the early form which represents the motion of a particle as a random motion in configuration space (the velocity is almost always infinite in this form) and in the later form which treats the motion as a random

motion in phase space. The probability interpretation at the atomic level of thermodynamical irreversibility is discussed in considerable detail. Finally, the recent statistical studies by the author and von Neumann of the gravitational field arising from a random distribution of stars are discussed. These studies of the gravitational force on a particle due to the rest of the system are still incomplete in that a full probability description of the moving system and of the varying forces has yet to be given. *J. L. Doob.*

Griffith, Laurence. A theory of the size distribution of particles in a comminuted system. Canadian J. Research. Sect. A. 21, 57–64 (1943). [MF 8404]

The author determines the distribution function for the size distribution of particles in a system which has been ground. The procedure is analogous to that used in statistical mechanics. Cf. a recent treatment of the same subject by A. Kolmogoroff [C. R. (Doklady) Acad. Sci. URSS (N.S.) 31, 99–101 (1941); these Rev. 3, 4]. *W. Feller.*

TOPOLOGY

Youngs, J. W. T. A note on separation axioms and their application in the theory of a locally connected topological space. Bull. Amer. Math. Soc. 49, 383–385 (1943). [MF 8382]

This note continues the discussion of hereditary classes of topological spaces initiated by G. E. Albert and the author in an earlier paper [Trans. Amer. Math. Soc. 51, 637–654 (1942); these Rev. 3, 314]. The author begins by defining a T-space as any topological space in which distinct points have closures with a degenerate product. He states that every T_1-space is a T-space and that every T-space is a T_0-space, and proves that the class of all locally connected T-spaces is hereditary. This result is of particular interest since in the previous paper it was established that the class of all locally connected T_0-spaces is an hereditary class while the class of all locally connected T_1-spaces does not enjoy this property. *D. W. Hall* (College Park, Md.).

Milgram, A. N. Some topologically invariant metric properties. Proc. Nat. Acad. Sci. U. S. A. 29, 193–195 (1943). [MF 8441]

"According to the well-known duality theorem of Alexander, each simple closed curve C in Cartesian n-space E_n is linked by an $n-2$ cycle Z in E_n-C. In the sequel it will be shown that Z can frequently be taken to be a metric $n-2$ dimensional sphere. A generalization will be formulated which holds not only in E_n, but in an arbitrary metric space."

The author applies the result to verify a conjecture by L. M. Blumenthal and to answer a question raised by I. J. Schoenberg. The conjecture was that every metric Peano continuum without an equilateral triple is an arc. [See Proc. Nat. Acad. Sci. U. S. A. 29, 107–109 (1943); these Rev. 4, 223 for the proof of this conjecture sketched by the reviewer).] In connection with his proof of the n-lattice theorem for metric arcs [Ann. of Math. (2) 41, 715–726 (1940); these Rev. 2, 130], Schoenberg asked whether each metric simple closed curve contains a simple closed equilateral (nondegenerate) polygon of n sides for every $n=3, 4, \cdots$. The author answers this in the affirmative. A slight difference in the problems of inscribing equilateral n-lattices in arcs and simple closed curves might be noted. For any metric arc B and any two distinct points p, q of B there is for every positive integer n an equilateral n-lattice in B with end-points p, q. On the other hand, easy examples show that it is not true that for each metric simple closed curve and each point p of the curve there is for every positive integer n a simple closed (nondegenerate) equilateral n-lattice of the curve which begins and ends at p. *L. M. Blumenthal* (Columbia, Mo.).

Peiser, Alfred M. Covering mappings. Duke Math. J. 10, 305–307 (1943). [MF 8470]

The covering spaces of a connected, locally connected and locally simply connected topological space K are defined by associated subgroups of the fundamental group F of K. A covering mapping is a self-mapping of a covering space, such that each point and its image are "over" the same point of K. Every covering mapping is shown to be generated by an element of F. Conditions are found to determine whether an element of F generates a covering mapping or homeomorphism for a given covering space. *S. S. Cairns* (Flushing, N. Y.).

Pontrjagin, L. Mappings of the three-dimensional sphere into an n-dimensional complex. C. R. (Doklady) Acad. Sci. URSS (N.S.) 34, 35–37 (1942). [MF 7594]

The paper contains a brief sketch (without any proofs) of a method for classifying the homotopy classes of the continuous mappings of the three-dimensional sphere into any complex with a trivial fundamental group. First the homologically trivial mappings are classified using a generalization of Hopf's linking invariant [cf. J. H. C. Whitehead, Ann. of Math. (2) 42, 409–428 (1941); these Rev. 2, 323]. To handle the homologically nontrivial mappings the author introduces a new product $u * v$ defined for two integral cochains u and v of dimensions r and s. The product $u * v$ is a cochain of dimension $r+s-1$ and satisfies the relation (mod 2):

$$\delta(u * v) = (\delta u * v) + (u * \delta v) + u \cup v + v \cup u,$$

where δ stands for the coboundary and $\cup$ for the "cup" product. *S. Eilenberg* (Ann Arbor, Mich.).

Erdös, P. and Kakutani, S. On non-denumerable graphs. Bull. Amer. Math. Soc. 49, 457–461 (1943). [MF 8398]

A graph is said to be complete if each pair of its vertices is joined by one and only one segment. The authors show that the complete graph with m vertices is expressible as the sum of a countable number of trees if and only if the cardinal number m is less than or equal to $\aleph_1$. They then derive as a consequence that the continuum hypothesis is true if and only if the set of all real numbers is decomposable into a countable number of subsets each consisting only of rationally independent numbers. *O. Frink.*

Alexandroff, P. On homological situation properties of complexes and closed sets. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 6, 227–282 (1942). (Russian. English summary) [MF 8357]

Given an abstract complex K and a closed subcomplex G, the author discusses the duality relations between the various relative homology and cohomology groups that may be considered [see S. Lefschetz, Algebraic Topology, Amer. Math. Soc. Colloquium Publ., vol. 27, New York, 1942, chap. 3; these Rev. 4, 84]. Using direct and inverse systems of groups the results are extended to locally bicompact

normal spaces. These results are applied to manifolds and Pontrjagin's generalization of the Alexander duality theorem is re-established. *S. Eilenberg.*

Clark, C. E. The Betti groups of the product of two normal spaces. Bull. Amer. Math. Soc. **49**, 307–313 (1943). [MF 8229]

For an open subset R of a normal space the inner cohomology groups $B^r(R)$ with integral coefficients were defined by P. Alexandroff [Trans. Amer. Math. Soc. **49**, 41–105 (1941), these Rev. **3**, 323]. The author shows that the cohomology group $B^r(R_1 \times R_2)$ of the Cartesian product $R_1 \times R_2$ contains a subgroup $B_1^r(R_1 \times R_2)$ such that both B_1^r and the factor group B^r/B_1^r are determined by the cohomology groups of R_1 and R_2. The group constructions are carried out using generators and relations. *S. Eilenberg.*

Clark, C. E. The Betti groups of symmetric and cyclic products. Bull. Amer. Math. Soc. **49**, 450–454 (1943). [MF 8396]

M. Richardson [Duke Math. J. **1**, 50–59 (1935)] has studied the homology groups of the symmetric products of a complex, while Walker [Bull. Amer. Math. Soc. **42**, 709–714 (1936)] did the same for the cyclic products. Using cohomology rather than homology the author is able to obtain more complete results. *S. Eilenberg.*

Montgomery, Deane and Samelson, Hans. Groups transitive on the n-dimensional torus. Bull. Amer. Math. Soc. **49**, 455–456 (1943). [MF 8397]

If a compact connected Lie group G is transitive and effective on a space E homeomorphic with an n-dimensional torus, then G is isomorphic with the n-dimensional toral group T_n and no element of G except the identity leaves any point of E fixed. The proof depends on expressing G in the form $(H' \times T_l)/N$, where H' is a compact connected simply connected Lie group and T_l is an l-dimensional toral group, and then, on the one hand, showing that the identity is the only subgroup of K which leaves fixed a point x of E, where K is the map of T_l in G, whence $l \leqq n$, and, on the other, mapping G on its orbit E so that the first homology group of G is mapped onto that of E, whence $l \geqq n$. If H is the map of H' in G, an argument involving the first homology group of the orbit $H(x)$ then shows that H contains only the identity and completes the proof. If E is an n-dimensional space with first Betti number n, the same proof applies and shows that E is a torus.

W. W. Flexner (Ithaca, N. Y.).

GEOMETRY

1945

***McCrea, William H. Analytical Geometry of Three Dimensions.** Oliver and Boyd, Edinburgh; Interscience Publishers, N. Y., 1942. vi+144 pp. $1.75.

***Castelnuovo, Guido. Lectures on Analytic Geometry.** Translated from the seventh Italian edition by Andrea Levialdi and Manuel Sadosky. Mundo Científico, La Plata, Argentina, 1943. 655 pp. (Spanish)

This is the first volume of a new series "Colección de Textos Científicos y Técnicos," published under the direction of M. Sadosky. It is a translation of the seventh Italian edition. The first edition appeared in 1903, the second in 1909. The following editions have been practically unchanged.

***Baker, H. F. An Introduction to Plane Geometry.** Cambridge University Press, Cambridge; Macmillan, New York, 1943. viii+382 pp. $4.00.

The scope of this book is most easily gauged from the table of contents: I: Euclid's theory of parallel lines; II: Propositions of incidence; III: The symbolic representation and Pappus' theorem; IV: Theorems proved from the propositions of incidence; V: The fundamental hypothesis; VI: The symbols of the real points of a line; VII: Involution and harmonic ranges; VIII: Related ranges and pencils; IX: Conics; X: Assignment of two absolute points, properties of circles; XI: The parabola; XII: The rectangular hyperbola; XIII–XIV: Theorems on conics; XV: Length and distance; XVI: Equation of conic and line; Notes. Approximately one third of the book is printed in a smaller type and is devoted to the statement of familiar theorems and to a valuable collection of exercises, some of which are attributed to J. H. Grace.

In the first six chapters Baker considers the foundations of Euclidean geometry and the introduction of a coordinate system; the approach is that of volume I of the author's Principles of Geometry [Cambridge University Press, 1922]. While necessarily abbreviated, these chapters are not so convincing as those that follow. An important feature of the book is the stress which is laid in chapter X on the arbitrariness in the choice of the two absolute points in the plane; this serves to emphasize those interesting generalizations of the centroid, the orthocenter and the circumcenter of a triangle, which make such a profound impression upon a thoughtful student. Baker's treatment of the conic is excellent, though his arguments would be easier to follow had there been more diagrams, a lack which is attributed to war conditions.

The book is to be highly recommended, particularly to the more serious students who will find it full of stimulating ideas. It is not easy reading at times, but the material in large print has been chosen with care and the reader can dip into the smaller print as his time and inclination allow.

G. de B. Robinson (Ottawa, Ont.).

Blumenthal, L. M. and Gillam, B. E. Distribution of points in n-space. Amer. Math. Monthly **50**, 181–185 (1943). [MF 8166]

This paper treats the geometrical interpretation of the signs of the cofactors of the elements of the Cayley-Menger determinant

$$D(p_0, p_1, \cdots, p_{n+1}) = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & p_0p_1^2 & \cdots & p_0p_{n+1}^2 \\ 1 & p_1p_0^2 & 0 & \cdots & p_1p_{n+1}^2 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & p_{n+1}p_0^2 & p_{n+1}p_1^2 & \cdots & 0 \end{vmatrix},$$

where p_ip_j denotes the distance of the points p_i, p_j in Euclidean space [see Menger, Math. Ann **100**, 75–163 (1928); Blumenthal, Distance Geometries, University of Missouri Studies, no. 13, 1938, in particular, pp. 56–59]. If the $n+2$ points $p_0, \cdots, p_{n+1}$ lie in an n-space E_n, then $D(p_0, p_1, \cdots, p_{n+1}) = 0$ [see Cayley, Collected Mathematical Papers, vol. 1, Cambridge University Press, 1889, pp. 1–4; Sylvester, Collected Mathematical Papers, vol. 1, Cambridge University Press, 1904, pp. 382–391], and the

classical theory interprets geometrically the signs of the cofactors of all elements in the principal diagonal of the Cayley-Menger determinant. This note completes the theory by determining the geometrical significance of the signs of the cofactors of the remaining elements of $D(p_0, p_1, \cdots, p_{n+1})$, the $n+2$ points being assumed to lie in E_n but not in an E_{n-1}. The results lead to a complete description of the distribution of $n+2$ points in E_n in terms of the mutual distances of the points. An application of the results is made in the proofs of two theorems known previously in special cases. Finally, statements are given of analogous theorems which have been proved for $(n+2)$-tuples in hyperbolic and spherical n-dimensional spaces. *G. B. Price.*

Blumenthal, L. M. Some imbedding theorems and characterization problems of distance geometry. Bull. Amer. Math. Soc. **49**, 321–338 (1943). [MF 8328]

This paper contains an exposition of known results, proofs of certain new theorems and statements of a number of unsolved problems in that part of metric geometry which deals with imbedding theorems and characterization problems. The author does not undertake to catalogue all results, but instead restricts himself to those with which he has had some connection. The first part of the paper presents new solutions of the imbedding problems known as the Euclidean subset problem and the Euclidean space problem, two problems which were first solved by Menger [Math. Ann. **100**, 75–163 (1928)]. The new solutions obtain the original results with weaker hypotheses. One of the improvements occurs in the theorem which establishes that E_n has congruence order $n+3$ with respect to the class of semimetric spaces; it permits the presence of certain free $(n+3)$-tuples, that is, sets of $n+3$ points which are not assumed imbeddable in E_n. The second part of the paper presents new results, the first of which relate to quasi-congruence order. Three theorems are proved which give conditions under which a semimetric space can be imbedded in E_n even when certain $(n+2)$-tuples are assumed free, that is, when certain $(n+2)$-tuples exist which are not assumed imbeddable in E_n. There follows a summary of efforts to express quasi-congruence order in terms of more elementary properties of the space [Blumenthal, L. M. and Robinson, C. V., Rep. Math. Colloquium (2), no. 2, 25–27 (1940); these Rev. **1**, 263] and statements of several unsolved problems. Next, there is a summary of results relating to congruence indices obtained by Blumenthal [Bull. Amer. Math. Soc. **47**, 435–443 (1941); these Rev. **3**, 138] and C. V. Robinson [Bull. Amer. Math. Soc. **47**, 818–819 (1941); these Rev. **3**, 89; Amer. J. Math. **64**, 260–272 (1942); these Rev. **3**, 300]. Finally, the author presents results on the imbedding of semimetric spaces in n-dimensional elliptic space and discusses certain related and unsolved problems. The paper deals with imbedding theorems and characterization problems based on global properties rather than local properties. *G. B. Price* (Lawrence, Kan.).

Prenowitz, Walter. Projective geometries as multigroups. Amer. J. Math. **65**, 235–256 (1943). [MF 8200]

The author develops projective geometry in terms of a single class G of undefined elements, "points" and one primitive notion "the join of a point and a point," with the aim of attaining "both the intuitive simplicity of the Veblen and Young approach and the elegance and generality of the algebraic methods of G. Birkhoff and Menger." In Menger's work "meet" and "join" are undefined operations. Here the meet of two figures is taken as the set-theoretic product, while the join of two sets is defined in terms of the join of two points. Denoting the latter by $a+b$, the postulates demand first that $a+b=b+a$ be a uniquely determined subset of G containing both a and b, with $a+a=a$. Defining $A+B$, for A, B nonnull subsets of G, as the set-theoretic sum $(a+b)$ extended over all a's in A and b's in B, it is supposed that $(a+b)+c=a+(b+c)$, and $a_1+a_2 \supset b \neq a_1$ implies $a_1+a_2=a_1+b$. The author observes that the system $(G, +)$ so formed is group-like in character and is a special case of structures studied under the names hypergroup or multigroup. The development of this system proceeds largely according to the pattern of Menger and G. Birkhoff, with the introduction of the notions of linear spaces, linear independence, dimension, etc. A bifurcation in the theory is produced by adjoining to the original system the postulate: $a+b$ consists of the points a and b, or its contrary. In the first case one is led to the Boolean algebra of all subsets of G, while the second alternative yields the classical projective geometries. The paper closes with a group-theoretic characterization of Boolean and projective groups among systems $(G, +)$. *L. M. Blumenthal.*

v. Sz. Nagy, Gyula. Zur Theorie der Flächen vom Maximalindex. J. Reine Angew. Math. **183**, 129–147 (1941). [MF 8645]

The order of a surface F is n provided there exists a line with exactly n points in common with F, while no line not contained in F has more than n points on F. A surface has index i if it has exactly i points in common with some line of projective space and each line meets it in at least i points. A smallest closed part of a surface which is itself a surface in the sense defined by the author is called a sheet (Schale). In this article the writer collects and furthers some investigations concerning surfaces of maximal index which he had previously published in the Hungarian language. The paper deals (for the most part) with questions concerning the existence of surfaces of maximal index with prescribed order and genus. Thus, for example, it is shown that there exist for each order $n \geqq 3$ one-sheeted ordinary surfaces of maximal index and genus 1, while for each even order n there are ordinary surfaces of maximal index and genus 0. For each order $n \geqq 3$, ordinary surfaces exist with maximal index and genus p $(1 \leqq p \leqq n-2)$, consisting of p sheets of genus 1. On the other hand, there is no surface of maximal index which consists of three or more ruled surfaces. *L. M. Blumenthal* (Columbia, Mo.).

Fraile, Arturo. Generalization of the ordinary analytic geometry. Revista Mat. Hisp.-Amer. (4) **2**, 285–294 (1942); **3**, 13–37 (1943). (Spanish) [MF 8353]

This paper expresses analytically the plane contour bounded by broken lines in one plane by means of a single equation; the idea is then extended to define whole areas within the plane. The idea was first presented by G. Tacchella [Giorn. Mat. (3) **71**, 1–52 (1933)], but the present paper is much more general. Consider

$$\varphi(x, y) = \sum a_i |f_i(x, y)| + \lambda F = 0,$$

in which F and each f_i are linear. The plane is divided into regions, each associated with a + or a − sign. Each point of a subregion so defined has the same sign. Each point of the boundary satisfies the equation. Each region is either a segment, a half-line, a point or is vacuous. The procedure applies to open or closed polygons, the sides of which may be straight or curved. Definite rules are given for defining such regions, with particular emphasis on the cases in which

each f_i is linear. Numerous examples are worked out in detail. The double modulus representation is of particular use in graphs of regions having a line or a point of symmetry. When all the f_i are concurrent, the equation represents a whole area. The concept is capable of various generalizations. *V. Snyder* (Ithaca, N. Y.).

García Rúa, J. On radial third-order cycles. Revista Mat. Hisp.-Amer. (4) **2**, 279–284 (1942). (Spanish) [MF 8351]

The author considers a tetrahedral angle $\alpha\alpha'\alpha''\delta$ and a homography φ such that $\alpha\alpha'\alpha''\delta \overline{\wedge} \alpha'\alpha''\alpha\delta$ and studies some elementary properties of the configuration under φ, φ^2, $\varphi^3=1$. *L. M. Blumenthal* (Columbia, Mo.).

Hohenberg, Fritz. Über die Kegelschnitte mit gemeinsamen Hauptachsen. Deutsche Math. **6**, 530–537 (1942). [MF 8601]

Given a conic, its equation expressed in the canonical form and an image point, the coordinates of which are the square roots of the major and minor axes. Various systems of conics are studied, the image curves of which are given. [Earlier related papers by the same author are Monatsh. Math. Phys. **50**, 111–124 (1941) and Akad. Wiss. Wien, S.-B. IIa. **150**, 89–108 (1941).] *V. Snyder.*

Sanguineti, Jeronimo. Some marginal notes. An. Soc. Ci. Argentina **133**, 461–465 (1942). (Spanish) [MF 8429]

The paper concerns three miscellaneous elementary facts that are easily proved. (I) If p is a prime then $a^{p-1}+(p-1)!\equiv 0 \pmod p$; this is also generalized to an arbitrary modulus. (II) In a regular polyhedron with triangular faces, twice the number of edges equals three times the number of faces. (III) The sum of the squares of the distances from the foci of an ellipse (or hyperbola) to an arbitrary tangent of the curve is constant. This is deduced from the well-known fact that the product of the distances concerned is a constant. *H. W. Brinkmann.*

Labra, Manuel. Computation of the sides of important podal triangles. Revista Soc. Cubana Ci. Fís. Mat. **1**, 48–57 (1942). (Spanish) [MF 8309]

Baidaff, B. I. The metric relation of Stewart and another generalization. Bol. Mat. **15**, 134–136 (1942). (Spanish) [MF 8306]

Lense, J. Bemerkung zu meinem Aufsatz "Die sphärische Trigonometrie in der sphärischen Astronomie". Astr. Nachr. **272**, 77 (1941). [MF 8575]

The previous paper appeared in Astr. Nachr. **271**, 121–132 (1941); these Rev. **3**, 87.

Convex Domains, Integral Geometry

Newman, M. H. A. On a string problem of Dirac. J. London Math. Soc. **17**, 173–177 (1942). [MF 8267]

Let A and B be two convex solids, connected by n parallel strings. Let A be given a single rotation through 2π about any axis passing through it. Then if $n\geqq 3$ the strings cannot be restored to their original positions if A and B are held fixed. The author states this problem as a modification and generalization of the string problem of Dirac, in which a pair of scissors are fastened to the four legs of a chair by two strings, one through each handle, and the scissors are then revolved through k revolutions in the plane of the blades. The twists can be removed from the string without moving the scissors or chair if k is even, but not if k is odd. To prove his theorem the author uses a modified form of Artin's theory of plaits. To form a plait, a set of n strings are connected between parallel horizontal supporting lines, so that a variable parallel between them crosses each string only once, and in no position contains more than one point where adjacent strings cross. The crossing of the rth string over the $(r+1)$st corresponds to the operator σ_r of the plait group. A rotation of one of the supporting lines through 2π corresponds to the operator $(\sigma_1\sigma_2\cdots\sigma_{n-1})^n$. By a mixed geometric and algebraic argument it is shown that for n odd (>1) this operator is of period 2, and this proves the theorem. *J. S. Frame* (East Lansing, Mich.).

John, Fritz. An inequality for convex bodies. Univ. Kentucky Research Club Bull., no. **8**, 8–11 (1942). [MF 8158]

Let R be a convex body in n-dimensional Euclidean space and $F(R)$ the family of convex bodies obtainable from R by a nondegenerate affine transformation; $d(R)$ is the minimum and $D(R)$ the maximum distance between two planes of support of R. The author proves the following theorem: there exists $R'\varepsilon F(R)$ such that $D(R')/d(R')\leqq\sqrt{n}$. This inequality is exact; if R is an n-dimensional cube, $\min D(R')/d(R')=\sqrt{n}$, R' in $F(R)$. It is first shown that the expression (volume of R')/$[d(R')]^n$ realizes a minimum for a certain $\bar{R}$ in $F(R)$. By the application of an infinitesimal transformation it is proved that $D(\bar{R})/d(\bar{R})\leqq\sqrt{n}$, which proves the result stated. *J. W. Green.*

Kasner, Edward. Note on non-Apollonian packing in space. Scripta Math. **9**, 26 (1943). [MF 8343]

Proof that in 3 or n-space any limited portion of volume V can be filled by a finite number of nonoverlapping spheres whose total volume exceeds $V-\epsilon$. *P. Franklin.*

Kasner, Edward, Comenetz, George and Wilkes, John. The covering of the plane by circles. Scripta Math. **9**, 19–25 (1943). [MF 8342]

An explicit method of filling any limited portion of the plane of area A by a finite number of nonoverlapping circles whose total area exceeds $A-\epsilon$ is described. *P. Franklin.*

Santaló, L. A. Integral formulas in Crofton's style on the sphere and some inequalities referring to spherical curves. Duke Math. J. **9**, 707–722 (1942). [MF 7924]

A closed convex curve on the unit sphere which intersects every great circle in at most two points is called convex. Such a curve K divides the sphere into two parts, one of which lies wholly in a hemisphere. The area F of this smaller part is called the area of K. Let L be the length of K. Let ω be the angle between the two great circles tangent to K through the point Ω. Then

$$(*) \qquad \int\omega d\Omega=\pi L-\pi F, \quad \int \sin\omega\, d\Omega=\pi L-\tfrac{1}{4}L^2.$$

Here the integration is extended over all the points Ω outside of K, identifying diametral points. [Since no tangent of K passes through a point that is diametral to an interior point of K, diametral points may be identified on the whole sphere. Thus the formulas (*) seem rather to belong to elliptic geometry.] Application of the duality principle to

(*) yields two more formulas. The author also derives formulas involving the lengths of the tangents from Ω to K.

The second part of his paper contains a spherical analogue of a formula of Hornich [Monatsh. Math. Phys. 47, 432–438 (1939); these Rev. 1, 158]. Let L be the length of a rectifiable curve on the unit sphere. Let F be the area of the domain of the points whose spherical distance from the curve is not greater than $\rho \leqq \pi/2$. Then $F \leqq 2L \sin \rho + 2\pi(1-\cos\rho)$. The conditions for equality are established. Finally, by considering suitable exterior and interior parallel curves, the author derives upper and lower bounds for the isoperimetric deficit $L^2+F^2-4\pi F$ of a closed convex spherical curve with a continuous radius of spherical curvature. *P. Scherk.*

Algebraic Geometry

Bompiani, E. Sul diagramma di Newton relativo ad una singolarità di una curva algebrica piana. Math. Z. 48, 1–3 (1942). [MF 8558]

The paper considers the nature of singularities of a plane algebraic curve. Suppose the associated Newton polygon of a given singularity is constructed. A side of this polygon may have points, called lacunae, the coordinates of which do not belong to the diagram. The paper shows that the necessary and sufficient condition that a side of the characteristic polygon through its highest vertex contain a lacuna is that the polar of a definite order have the side of the polygon through it as a component of a certain multiplicity. *V. Snyder* (Ithaca, N. Y.).

Coolidge, Julian L. Systems of rational curves. Amer. J. Math. 65, 257–262 (1943). [MF 8201]

Let the homogeneous coordinates of a point P be given as polynomials of degree n in a parameter t, whose coefficients are analytic functions of a parameter v. The locus of P for a fixed v is a rational curve. The locus of P for variable v and t is an analytic surface. This paper discusses relations between the curves and the surface on which they lie. Among theorems stated we cite a few representative ones. If a one-parameter family of curves have the property that the inflections and the points where the osculating plane is tangent to the surface are all points where asymptotic directions coincide, then these curves will be projectively related by their conjugates on the surface generated. A one-parameter family of rational plane curves of order n will be projectively related by their conjugates on the surface generated if and only if their inflections are at points where asymptotic directions coincide, and they are tangent to n curves, or if it is a limiting form of such a system. Butel's [Ann. École Norm. (3) 7, 155 (1890)] theorem is a special case ($n=2$) of this theorem. If the rational curves lie on a nonplanar developable, they are projectively related by the generators. Finally, the rational curves of a two-parameter family of curves can be assembled in two different ways into surfaces where they are cut projectively by their conjugates, unless their tangents are the tangents to the asymptotic lines of a one-parameter family of surfaces, or a limiting case of such a set. *V. G. Grove.*

Moody, Ethel I. Notes on the Bertini involution. Bull. Amer. Math. Soc. 49, 433–436 (1943). [MF 8392]

A Bertini involution is defined by a pencil of cubic curves. A curve of the pencil is determined by a point y. The tangent at a base point B to this curve meets it again at R. The line Ry meets it again at y'. The Bertini involution associated with B transforms y into y'. In the present note the equations of the transformation are obtained in reduced form without extraneous factors. The product B_1B_2 of two transformations is not periodic. It leaves three cubics of the pencil point by point invariant. The product of three different Bertini involutions associated with the same pencil of cubic curves is involutorial. *V. Snyder.*

Godeaux, L. On surfaces of fourth order which contain two skew cubics. Revista Mat. Hisp.-Amer. (4) 3, 4–12 (1943). (Spanish) [MF 8352]

About forty years ago Severi [Rend. Circ. Mat. Palermo 30, 265–293 (1910) contains the references to the earlier papers] wrote a series of papers on the theory of the base, expressing all the curves on a given algebraic surface as a linear function of multiples of the curves of the base. The scheme is particularly useful in the study of the birational transformations which leave the surface invariant. This was done by Fano [who published nine notes on this theme in 1922 in Accad. Naz. Lincei. Rend.] for various quartic surfaces and by the present author [Acad. Roy. Belgique. Bull. Cl. Sci. (5) 29 (1942)] for the quartic surface containing a straight line and a conic. The present paper considers a quartic surface through two skew cubic curves having no point in common. The base consists of these two cubics and a plane section. The genera of the surface are $p_a=P_4=1$. The bisecants of the cubic k_1 meet the given surface F in a pair of points conjugate in an involution T_1, similarly for T_2. The product T_1T_2 is not periodic. These transformations are then applied to the study of systems of elliptic quartics, two such curves being the complete intersection of F and a quadric. Various properties of cubic surfaces are derived and their plane representations obtained by the same process. By taking for base a plane section, one of the cubics and the quintic residual of a quadric surface through the other with F, another system of involutorial transformations is obtained. *V. Snyder.*

Godeaux, Lucien. Sur la surface du quatrième ordre contenant trente-deux droites. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 25, 539–552 (1939). [MF 8642]

Rozet, O. Sur la construction d'une surface projectivement canonique. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 25, 670–673 (1939). [MF 8644]

Rozet, O. Sur les surfaces de genres un, d'ordre huit, de l'espace à cinq dimensions. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 25, 582–587 (1939). [MF 8643]

Gauthier, L. Une involution d'ordre deux représentant la variété cubique de l'espace à quatre dimensions. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 25, 518–529 (1939). [MF 8641]

Parts I and III appeared in the same Bull. 25, 375–381 (1939); 27, 569–578 (1941); these Rev. 2, 14; 3, 303.

Srinivasiengar, C. N. On the quartic developable. J. Indian Math. Soc. (N.S.) 6, 127–130 (1942). [MF 8295]

This paper discusses the tangent developable of the twisted cubic from the projective point of view. A typical theorem may be stated as follows. The first polar surfaces of points on a twisted cubic with respect to its tangent developable form a family of Cayley cubic scrolls and the

generators of the quartic developable belong to the same linear complex. Each generator of any one of the Cayley scrolls intersects the twisted cubic and lies in the osculating plane of the cubic at that point. *V. G. Grove.*

Edge, W. L. Notes on a net of quadric surfaces. V. The pentahedral net. Proc. London Math. Soc. (2) **47**, 455–480 (1942). [MF 8271]

Four earlier papers by the same author, the last of which [Proc. London Math. Soc. (2) **47**, 123–141 (1941); these Rev. **3**, 15] contains the references to the earlier ones, dealt for the most part with general nets of quadric surfaces in [3]. The present note is concerned with a special net having a self conjugate pentahedron. This condition is poristic; the existence of one such pentahedron involves the existence of a singly-infinite set [Reye, J. Reine Angew. Math. **82**, 75–94 (1887)]. The faces of the pentahedra all belong to a developable ω of the third class, osculating a space cubic curve γ. The sets of osculating planes belong to a g_3'. A plane of ω may be defined by

$$P \equiv x_0 + \theta x_1 + \theta^2 x_2 + \theta^3 x_3.$$

Given two sets of five values of θ, each defined by a quintic equation $(\theta)=0$, $g(\theta)=0$, with constant coefficients; the g_3' is given by the pencil $\lambda f(\theta)+\mu g(\theta)=0$. A net of quadrics is given as functions of θ. The Jacobian curve ϑ, a sextic which is the locus of the vertices of the cones in the net, is then considered, including the ruled surface R^8 of its trisecants and various apolarity properties. A (1, 1) correspondence exists between the quadrics of the net and the axes of the developable F_4; R^8 and F_4 have eight common generators.

In the second part, lines of [3] are represented by the points of a quadric primal Ω in [5]; the generators of R^8 are represented by the points in which Ω is met by a Veronese surface, but in the present case this lies on Ω. In the third part the net is still further specialized; the pencil of binary forms is now $\lambda\theta+\mu\theta=0$. The net is now defined by

$$Q_0 \equiv x_2^2 + 2x_1x_3, \quad Q_1 \equiv 2x_0x_3 + 2x_1x_3, \quad Q_2 \equiv x_1^2 + 2x_0x_2.$$

The base consists of X_0 and X_3 of the reference tetrahedron, each taken twice. The Jacobian curve consists of two space cubics which touch at each base point; R^8 is composed of two ruled quartics, having γ_1, γ_2 for double curves, and each containing the other simply. *V. Snyder.*

***Segre, B. The Non-singular Cubic Surfaces.** Oxford University Press, Oxford, 1942. xi+180 pp. $4.25.

This monograph contains a most exhaustive study of non-singular cubic surfaces, especially in the real domain. As the author tells us in the preface, his object is to give a complete theory of what Cayley has called "the complicated and many-sided symmetry" of the relations between the 27 lines of a cubic surface. Such a theory naturally entails a minute analysis of an endless row of details, each of which represents a feature of the complicated pattern of the above-mentioned symmetry relations. This emphasis on details, which is inherent to the subject matter, and the methodical way in which the author goes about his object of exhausting the properties of a cubic surface may frighten away some readers who prefer in mathematics a better balance between the general and the particular, between the abstract and the concrete. But the reader who is willing, so to speak, to live for a while on a cubic surface and to read the book in the spirit in which it has been written will be greatly rewarded by the elegance and ingenuity of the author's treatment of the subject. It is based on the principle of continuity and the method of degeneration. The nonsingular cubic surface F is made to vary continuously, say in a pencil, until it degenerates into a cubic surface F_0 consisting of three independent planes π_i. These planes meet in three lines r_i, and it is permissible to assume that the intersections of r_i with the variable F have definite limits P_{i1}, P_{i2}, P_{i3}. It is shown that the 27 lines of F tend to the lines joining pairs of points $P_{\alpha_1\beta}P_{\alpha_2\gamma}$ lying on distinct lines ($\alpha_1 \neq \alpha_2$). We thus have a graphical representation of the 27 lines of F by their images on F_0, a representation which can be pictured in the plane by projecting the three lines r_i onto a plane. At the same time a convenient notation is introduced whereby the line of F whose image on F_0 is the line joining $P_{\alpha_1\beta}$ and $P_{\alpha_2\gamma}$ is denoted by an ordered triple of integers, one of which is zero, while the other two are equal to β and γ and have respectively the α_1th and α_2th place.

The incidence relations between the 27 lines are readily analyzed by means of the graphical representation, in virtue of the following property: two lines on F are incident if and only if their images (that is, their limits) on F_0 are lines which either lie in the same plane π_i or contain the same point $P_{\alpha\beta}$, these two circumstances, however, not occurring at the same time. As an illustration, let us consider the two most important configurations of lines on F: the double-sixes and the Steiner sets.

A double six consists of two complementary sextuplets of skew lines such that each line of one sextuplet is skew to one and only one corresponding line of the other sextuplet. The graphical representation shows then immediately that a double-six can be represented on F_0 only by two figures: (1) one figure consists of the 12 lines passing through two distinct points of a line r_i; (2) the other figure consists of the 12 lines through three points of different lines r_1, r_2, r_3, with the exception of the sides of the triangle determined by these points. There are 9 figures of the first type and 27 of the second type, giving altogether the 36 double-sixes of F. It is to be noted, however, that these two types of figures are incidental to a specific way of continuously deforming F into F_0 and do not correspond to an intrinsic difference between double-sixes on F. This graphical representation of the double-sixes leads to the following notation: (1) a double-six of the first type is denoted by $\{\alpha\beta\}$ if the two points lie on r_α and are distinct from the point $P_{\alpha\beta}$; (2) a double-six of the second type is denoted by $\{\alpha\beta\gamma\}$ if the three points are $P_{1\alpha}$, $P_{2\beta}$, $P_{3\gamma}$.

A Steiner set of 9 lines consists of three associated triplets of skew lines such that each line of one triplet is incident with two and only two lines of each other triplet. The graphical representation of a Steiner set can be of three different types. One of these types is very simple to describe; it is the figure formed by the 9 lines in a plane π_i. This figure exhibits very clearly the fact that there are two ways of splitting a Steiner set into three associated triplets for, say, in the plane π_1 we have not only the triplets $O1i$, $O2i$, $O3i$, but also the triplets $O1i$, $O2i$, $O3i$ ($i=1, 2, 3$).

These and similar considerations form the substance of the first chapter which contains a thorough study of the 27 lines in the complex domain. Here, as elsewhere in this book, a large number of carefully drawn figures, pertaining to the graphic representation, enables the reader to visualize in the plane the various configurations of the 27 lines on F.

But the most original and novel application of the method of degeneration is made by the author in the third and largest chapter which deals with real nonsingular cubic

surfaces. It is first observed that, as a real cubic surface F tends (in the real domain) to a surface Φ having a real node O, one double-six of F approaches the set of 6 lines of Φ through O. This happens in such a fashion that corresponding lines of the double-six approach one and the same line through O. The combination of two features determines the type of the real surface F: (1) the number of real lines of Φ through the node O; (2) the property of the two complementary sextuplets of the double-six of being either self-conjugate or mutually conjugate. This leads to 5 possible types F_i of real cubic surfaces, where F_1, F_2 and F_3 contain, respectively, 27, 15 and 7 real lines, while both F_4 and F_5 contain only 3 real lines, but differ in the number of complex lines of first and second kind (a complex line is of the first kind or of the second kind according as it does or does not meet its conjugate line): F_4 carries 12 lines of each kind, while all the 24 complex lines of F_5 are of the first kind. It is shown that the surfaces of the same type actually form a connected continuous system in the real domain. In the study of these 5 types of real cubic surfaces, an important role is played by the classification of the real lines on the surface into 4 categories. In the first place a real line r is called hyperbolic or elliptic according to the type of involution cut out on the line by the conics which are residual intersections of the surface F with the planes through r. An hyperbolic line r can be of two kinds according as there does or there does not exist a plane through r having no further real intersection with the surface. Elliptic lines r are classified into left-handed and right-handed lines, for, as a point P moves on an elliptic line r in one direction, the tangent plane at P may turn around r in one of two possible directions. Each of the 5 types of real cubic surfaces is thoroughly studied as to the lines of various types which exist on it, and also for questions of reality concerning its double-sixes, Steiner sets, etc. All these questions are analyzed and interpreted in the light of the graphical representation of the 27 lines. The various possibilities concerning the real elements of this representation reflect differences between the 5 types of surfaces F_i. Thus each F_i, $i \neq 5$, can be degenerated into 3 real planes π_i, but for F_5 only one of the planes π_i can be real.

The chapter on real cubic surfaces contains also an interesting study of the topology of each F_i. It is found that F_i, $i \neq 4$, is a closed connected manifold, while F_5 consists of two connected pieces, one of which is ovoidal (whence the real lines of F_5 lie on the second, nonavoidal piece). F_i, $i \neq 4$, and the nonovoidal part of F_5 are decomposed into cells by the real line of the surface, this decomposition giving rise to a generalized polyhedron. The cells of these polyhedrons are carefully studied as to the number of their faces, the types of the real lines on their boundaries, their mutual incidence relations, etc. Each cubic surface F_i (except naturally the ovoidal part of F_5) is a nonorientable manifold. Their genera are 7, 5, 3 for $i = 1, 2, 3$, respectively, while F_4 and the nonovoidal part of F_5 are homeomorphic to the real projective plane.

The group of the 27 lines is thoroughly studied both in the complex and in the real domain. In the complex domain the order $(3!)^4 \cdot 40$ and some structure properties of this group G are readily derived by analyzing the effect of the elements on G on the 40 triads of complementary Steiner sets or on the 36 double sixes. Here the graphical representation of the 27 lines is very helpful. Of particular interest is the definition of G as an abstract group defined by a set of generators and generating relations. For each double-six the group G contains an involutorial transformation σ which interchanges corresponding lines of the complementary sextuplets of the double-six and leaves invariant each of the remaining 15 lines. We thus find 36 involutary transformations in G. It is shown that they are generators of G, and also a complete set of generating relations between them is derived. Among these we point out the permutability relations which have a direct geometric significance: two transformations σ_1, σ_2 are permutable if the corresponding double-sixes have 4 lines in common. If, however, this is not the case, then the two double-sixes have 6 lines in common, and the remaining 12 noncommon lines form a third double-six, the corresponding involutorial transformation being $\sigma_1^{-1}\sigma_2\sigma_1$. The group G of the 27 lines can be realized by a continuous variation of F in a pencil, provided (1) the singular surfaces of the pencil are uninodal and (2) the ruled surface generated by the 27 lines of the variable F is irreducible.

In the real domain each surface F_i of the 5 types discussed above has a group G_i which is of course a subgroup of G. These 5 groups G_i are studied and are explicitly determined.

We have reviewed here briefly the contents of the first three chapters which bear, respectively, the following titles: (I) The 27 lines and one of their simplest degenerations; (II) The group G of the 27 lines; (III) The real cubic surfaces. This third chapter contains also a classification of the real double-sixes and a classification of the real cubic surfaces in the affine space. The fourth (last) chapter deals with miscellaneous topics which, for lack of space, we shall only list: the Sylvester pentahedron; the autoprojective cubic surfaces; the deduction of the type of a real non-singular cubic surface from its canonical (Sylvester) equation. There are three brief appendices. *O. Zariski.*

Segre, B. On limits of algebraic varieties, in particular of their intersections and tangential forms. Proc. London Math. Soc. (2) **47**, 351–403 (1942). [MF 7408]

As the method of degeneration is frequently used in algebraic geometry, the author undertakes a general study of the limiting processes on which that method depends. In the first two sections he gives some definitions and general properties. These refer to intersections of algebraic varieties and to the limit, as λ approaches λ_0, of an algebraic variety $M(\lambda)$ depending on a continuous parameter λ. The ambient space is always an algebraic variety V free from singularities. A distinction is made between the general limit and the regular limit, according as the limit depends or does not depend on the mode in which λ approaches λ_0. The general limit is merely the set-theoretic sum of all possible limiting positions $M(\lambda)$ and its dimension is never less than the dimension of $M(\lambda)$. Dealing with the intersection (MN) of two varieties on V, the multiplicity of each component of (MN) is defined as in Severi [while the case of a component of regular dimension has been set on a solid algebraic basis by van der Waerden, the reviewer is not certain about the state of affairs in the case of components of irregular dimension]. The main problem is formulated as follows.

Suppose that the intersection (MN) of a variable pair of varieties $M(\lambda)$, $N(\lambda)$ is of regular dimension d, but that the intersection (M_0N_0) of the limiting pair is $\Delta+\Gamma$, where Δ is of dimension d and Γ is of a higher dimension. If the regular limit L of (MN) exists, it will be exactly of dimension d: $L=\Delta+\gamma$. The d-dimensional variety γ is called the geometrical contribution of Γ. For different modes of approach $M\to M_0$, $N\to N_0$, γ may vary but, under certain general hypotheses, γ will remain within a system of equiva-

lence on Γ. This system of equivalence is called the functional contribution of Γ and its determination is the main problem. This problem is solved completely in the present paper in the special case in which M and N are curves on an algebraic surface: if M_0 and N_0 have a component in common, say $M_0=\Gamma+C$, $N_0=\Gamma+C'$, then $\Delta=(CC')$, and the functional contribution is given by the equivalence $\gamma(\Gamma, C+C'+\Gamma)$ which holds on Γ. This result is used for an analysis of an example considered by Enriques in connection with a proof of the completeness of the characteristic series [Accad. Naz. Lincei. Rend. (6) **27**, 493–498 (1938)] and is also used in order to clear up a deduction made by the author elsewhere in the same connection [Ann. Mat. Pura Appl. (4) **17**, 107–126 (1938)]. This analysis shows quite generally that the linear series cut out on a variable curve $A(\lambda)$ by the curves of a variable linear system $|A'(\lambda)|$ (for instance, by the adjoint curves of $A(\lambda)$) need not have as general limit the linear series cut out on the limit curve A_0 by the limit system $|A_0'|$. The latter series is always a part of the general limit, but, if the general limit of $A'(\lambda)$ contains reducible curves having A_0 as a component, there may exist limiting sets of points on A_0 which are not cut out by curves A_0'. The above result concerning pairs of curves on a surface is partially generalized to the case of t variable primals on a t-dimensional variety.

The rest of the paper deals mainly with the problem of determining the limit of the system of tangent hyperplanes of a variable hypersurface. This question is studied, more generally, in the case of analytical systems of analytical hypersurfaces. The considerations involved are mainly of a local character and the conclusions reached are too involved to be stated concisely in the present review. As an illustration of the type of results obtained we quote the following result: if an hypersurface A varies in a pencil and degenerates into $r_1A_1+r_2A_2+\cdots+r_sA_s$, then the system of tangent hyperplanes of A tends to the sum of several systems of hyperplanes, each counted to a certain (calculated) multiplicity. The individual systems are given by the hyperplanes which are either tangent to the intersection of the hypersurfaces A_i, taken h by h, or are tangent to the intersection of A with the A_i, again taken h by h ($h=1, 2, \cdots$).

O. Zariski (Baltimore, Md.).

Differential Geometry

DeCicco, John. New proofs of the theorems of Beltrami and Kasner on linear families. Bull. Amer. Math. Soc. **49**, 407–412 (1943). [MF 8388]

New proofs, from a uniform point of view, of Kasner's theorem characterizing isothermal families, Kasner's theorem characterizing surfaces of constant curvature in terms of isothermal families and Beltrami's theorem on the geodesic mapping of surfaces of constant curvature on a plane.

P. Franklin (Cambridge, Mass.).

Bortolotti, Enea. Über die Invarianten von Linienelementen und die projektive Geometrie der Kurvengewebe. Monatsh. Math. Phys. **50**, 257–281 (1942). [MF 8545]

This paper is an exposition of the subject of the invariants of lineal elements and the projective geometry of webs (and nets) of curves. An exhaustive bibliography of 92 papers on this and kindred subjects is appended. *V. G. Grove.*

Terracini, Alejandro. Contributions to the geometric study of the differential equations $y'''=F(x,y,y')+G(x,y,y')y''+H(x,y,y')y''^2$. Univ. Nac. Tucumán. Revista A. **3**, 195–234 (1942). (Spanish) [MF 8152]

Kasner [Differential-geometric Aspects of Dynamics, Amer. Math. Soc. Colloquium Publications, vol. 3, part II, New York, 1913 (reprinted 1934)] showed that, under any point transformation from a surface S upon a plane π, the ∞^3 dynamical trajectories of a field of force upon S are represented by a differential equation of the form

$$(F)\qquad y'''=F(x,y,y')+G(x,y,y')y''+H(x,y,y')y''^2.$$

The study of the dynamical trajectories of a surface S is invariant under the group of geodesic preserving point transformations. If $F(x,y,y')$ is identically zero, there result systems of ∞^3 curves of the type (G). Many problems of geometry and physics give rise to equations of the type (G). [See Kasner, Proc. Nat. Acad. Sci. U. S. A. **28**, 333–338 (1942); these Rev. **4**, 54.] Of special importance are the dynamical trajectories of a field of force in the plane. Kasner obtained a completely characteristic set of five (metric) properties of the trajectories, and the author has transformed these into purely projective language [Univ. Nac. Tucumán. Revista A. **2**, 245–329 (1941); these Rev. **4**, 54].

The type (F) is invariant under the group of all point transformations whereas (G) is invariant under the projective group. Nevertheless two geometric characterizations of the type (F) are obtained in purely projective language. The first one is obtained by constructing an auxiliary projective space σ_3. The second one is entirely in the original plane although the proof is based upon the associated three-space σ_3. It may be described in the following manner. At any lineal element $E_1(Aa)$ consisting of the point A and the line a through A, construct the ∞^2 conics which have third order contact with the ∞^1 curves of any system Σ of ∞^3 curves passing through E_1. These conics may be separated into ∞^1 sets of ∞^1 conics such that the conics of any such set have contact at E_1 with all the ∞^1 curves of Σ through E_1, and such that they have second order contact with their envelope Δ. Thus there is determined a family S_Δ of ∞^1 curves corresponding to the lineal element $E_1(Aa)$. The system Σ is of the type (F) if and only if the family S_Δ consists of ∞^1 cubical curves Δ, each one of which has A as a point of inflection and a as inflectional tangent and also possesses a cusp, which describes a line through A as Δ varies

In the second part the author considers systems of ∞^3 curves of the type (F) defined upon an arbitrary surface S. Some examples of the type (F) upon a (nondevelopable) surface S may be obtained by placing a certain projective differential invariant I of Bompiani [Rend. Sem. Mat. Fis. Milano **10**, 9–36 (1936)] equal to a constant k. The resulting systems of ∞^3 curves Σ_k are all of the type (F). For $k=0$, the system Σ_0 is a sectional family [Kasner, Proc. Nat. Acad. Sci. U. S. A. **17**, 370–376 (1931)], and for $k=-1$ the system Σ_{-1} represents the projective lines on S of Segre. Finally the author considers when any set of ∞^1 third order differential elements of the type (F) containing a lineal element E_1 can be projected from a point Z into a set of ∞^1 third order elements of the type (G). There are in general two (principal) planes passing through the tangent line a of E_1, which constitute the bases for the centers of projectivity (G) with respect to E_1. A dual of this result is obtained giving two (principal) points upon the tangent line a^* conjugate to the tangent line a.

Many other topics are studied, among which may be mentioned the relations between the principal planes and points, special forms of systems of the type (F), a new characterization of the projective curves of Segre and systems of the type (F) which are projectively related to systems of the type (G). *J. DeCicco* (Chicago, Ill.).

Santaló, L. A. On the concept of curvature of a surface. Math. Notae 2, 165–184 (1942). (Spanish) [MF 7946]

Levi, B. Brief comment. Math. Notae 2, 184–187 (1942). (Spanish) [MF 7947]

This paper contains geometric interpretations of the Gaussian curvature of a surface, of its mean curvature and of other symmetric functions of the two principal curvatures. In particular, it contains a slight generalization of Bertrand's and Diguet's formulas and proofs of some related formulas of Blaschke [cf. Differentialgeometrie, vol. 1, 3rd ed., Springer, Berlin, p. 153, formula (33) and p. 120, formulas (147), (148)].

Remarks by B. Levi concerning the Gaussian curvature and the metric of the imbedding space are added.

P. Scherk (Bloomington, Ind.).

Behari, Ram. Ruled surfaces whose curved asymptotic lines can be determined by quadratures. Math. Student 10, 59–63 (1942). [MF 7952]

The author obtains a number of cases in which the curved asymptotic lines of a ruled surface can be obtained by quadratures. The system of curved asymptotic lines is represented by a Riccati equation, which in special cases is solvable in terms of quadratures. Among these cases are: (1) the generators of the ruled surface remain parallel to a fixed plane; (2) the generators are perpendicular to the binormals to the directrix curve. *J. W. Green.*

Hsiung, Chuan-Chih. Asymptotic ruled surfaces. Duke Math. J. 10, 217–237 (1943). [MF 8462]

Let the parametric curves on a surface S be chosen as the asymptotic curves. Let Σ_u, Σ_v be the asymptotic ruled surfaces of S at a point x on S. The author shows that the curve of section of Σ_u by a plane through the u-tangent has a point of inflexion at x whose point O_4 (of Bompiani)[Boll. Un. Mat. Ital. 5, 118–120 (1926)] is projective to the plane of the section. If the first directrix of Wilczynski intersects the quadric Q of Lie in a point M_3 and the second directrix intersects the v-tangent in M_2, then a plane section of Σ_u by a plane through M_2M_3 has an inflexion at M_2 whose Bompiani point O_4 is projectively related to the plane of the section. Let the given planes intersect in l and denote by l' the line joining the two points O_4 of Bompiani so described. Then as l' generates Q, l generates a quadric $Q_1^{(n)}$, and as l generates Q, l' generates a quadric $Q_{-1}^{(n)}$. By replacing Q by first $Q_1^{(n)}$, and then $Q_{-1}^{(n)}$, and so on, there is generated a sequence $\{Q_i^{(n)}\}$ of quadrics. By interchanging the roles of the asymptotic curves a second sequence $\{Q_i'^{(n)}\}$ is generated. Among the properties of these quadrics we cite the following. Any two quadrics $Q_i^{(n)}$, $Q_i'^{(n)}$ of the sequences intersect in the asymptotic tangents and in a conic the pole of whose plane with respect to Q is a point on the second directrix of Wilczynski.

Let Q_u and Q_v be the Moutard quadrics of Σ_u, and Σ_v determined by a nonasymptotic tangent t to S at x. These quadrics intersect in the asymptotic tangents and in a conic the plane of which intersects the tangent plane to S at x in the tangent t if and only if t is a Segre tangent. The three planes, in which lie the residual conics of intersection of the Moutard quadrics of Σ_u, Σ_v determined by the three Darboux tangents, intersect in a certain canonical line of the first kind. Again the quadrics Q_u, Q_v each intersect a quadric of Darboux (not Q) in a residual conic the plane of each of which envelopes a cone of third class the cusp-axis of which is a scroll directrix of Sullivan. If the quadric of Darboux is the quadric of Lie these cones are quadric cones. The polar planes of the asymptotic tangents (arranged properly) with respect to these quadric cones intersect in the first edge of Green. Let π be any plane distinct from the tangent plane. The plane π intersects the asymptotic ruled surfaces in curves having ordinarily second order contact at x. If they have contact of the third order, then π must pass through a tangent of Segre. Finally generalization of the transformation of Čech is made, as well as a new construction for the correspondence of Palozzi.

V. G. Grove (East Lansing, Mich.).

Chang, Su-Cheng. On the quadric of Lie. Bull. Amer. Math. Soc. 49, 257–261 (1943). [MF 8221]

The characteristic curve C of the quadric Q of Lie associated with an ordinary point P of a nonruled nondegenerate surface F is the limiting intersection of Q and another quadric of Lie moving into coincidence with Q. The author shows that there are on F a one-parameter system of curves L along each of which C degenerates into two conics. He also proves some theorems concerning these conics.

T. R. Hollcroft (Aurora, N. Y.).

Anas, Mehmet. Surfaces dont le second Beltramien relatif à la courbure moyenne est nul. Rev. Fac. Sci. Univ. Istanbul. Ser. A. 6, 154–188 (1941). (French. Turkish summary) [MF 8453]

This paper considers surfaces for which the second differential parameter of the mean curvatures J of the surfaces vanish. Developable surfaces, including cones and cylinders, and surfaces of revolution are considered. A typical theorem may be stated as follows: if the second differential parameter of the mean curvature vanishes, and the curves $J=$const. coincide with one of the families of curves forming the lines of curvature, the surface is a surface of revolution. The paper concludes by showing that all of the invariants of the surfaces under discussion are expressible in terms of J, the first differential parameter of J and another first order vector differential parameter of J. *V. G. Grove.*

Schaaff, Wilhelm. Bemerkungen über eine Arbeit von Heinz Schröder: Über die Verbiegung der Flächen zweiter Ordnung. Jber. Deutsch. Math. Verein. 52, 59–60 (1942). [MF 8533]

Schröder's paper appeared in the same Jber. 47, 63–68 (1937).

Wagner, V. On the Cartan group of holonomicity for surfaces. C. R. (Doklady) Acad. Sci. URSS (N.S.) 37, 6–8 (1942). [MF 8494]

The tangent planes along a curve on a surface in Euclidean space are each referred to orthogonal coordinates regarded as components of a complex variable z. Thereby the usual differential equations for Euclidean point displacements become a single linear equation in z whose solutions are immediately obtained in terms of quadratures. Closing the curve of integration furnishes a finite transformation of the holonomy group in the tangent plane at

the initial point P. This is resolved into a translation and rotation, the former depending only on the line integral of geodesic curvature, the latter on the surface integral of the Gaussian curvature. By the use of geodesic polar coordinates at P the translation is shown to be (within infinitesimals of third order) perpendicular to the radius vector of the centroid of a map in the tangent plane of the area enclosed by the curve. *J. L. Vanderslice* (College Park, Md.).

Bell, P. O. and Foreman, W. C. Euclidean applications of the projective differential geometry of the R_λ-correspondent. Ann. of Math. (2) **44**, 298–314 (1943). [MF 8286]

This paper deals with various aspects of the projective and Euclidean differential geometry of an arbitrary net of curves on a surface S in 3-space. The authors choose parameters on S so that the arbitrary net is parametric and associate with each tangent direction λ at a point of S another tangent direction ξ which they call the R_λ-correspondent. This correspondence between λ and ξ at a point is defined with respect to the parametric net and is an involution. The double elements of the involution form a net of integral curves called the principal associate net of the parametric net. Various properties of the principal associate net and of other nets (defined intrinsically by means of R_λ-correspondence) are discussed from the standpoint of projective differential geometry. If the Euclidean geometry of S is considered, the R_λ-correspondence between λ and ξ is defined in terms of the coefficients of the second fundamental form of S. A dual correspondence is defined by replacing the second fundamental form in this definition by the first fundamental form of S. Geometric characterizations of these dual correspondences are obtained. These characterizations are used in the formulation of dual theories in the Euclidean differential geometry of nets on S. The net consisting of the curves whose normal curvatures at a point are equal to the mean normal curvature of S at the point and the dual of this net play important roles in these theories. New characteristic properties of the lines of curvature and of the curves of Segre and of Darboux are obtained. *A. Fialkow* (New York, N. Y.).

Coburn, N. Conformal geometry of unitary space. Univ. Nac. Tucumán. Revista A. **3**, 125–140 (1942). [MF 8148]

This paper deals with the conformal geometry of a unitary space of n dimensions. An equi-conformal fundamental tensor is introduced which leads to an equi-conformal connection, a conformal connection and certain conformal curvature affinors. Necessary and sufficient conditions are given for the existence of conformal fundamental tensors. Finally it is shown that no unitary space $K_n (n>2)$ of constant nonvanishing curvature is a conformal zero unitary space (space for which the conformal curvature affinor vanishes) and it is pointed out that this result is in exact opposition to the corresponding theorem for Riemannian space. *T. Y. Thomas* (Los Angeles, Calif.).

Wong, Yung-Chow. Some Einstein spaces with conformally separable fundamental tensors. Trans. Amer. Math. Soc. **53**, 157–194 (1943). [MF 8118]

When the fundamental form of a Riemannian manifold V_m with x^α, $\alpha, \beta = 1, 2, \cdots, m$, can be written in the form

$$ds^2 = \rho^{-2} g_{ij} dx^i dx^j + \sigma^{-2} g_{pq} dx^p dx^q = {}^a g_{\alpha\beta} dx^\alpha dx^\beta,$$
$$i, j = 1, \cdots, n; \quad p, q = n+1, \cdots, m;$$

where ρ and σ depend on all x^α, but g_{ij} only on x^i and g_{pq} only on x^r, the $V_m (m>2)$ is called conformally separable of the type $(n, m-n)$. The construction of such V_m, which are at the same time Einstein spaces E_m, leads to the study of the equation

$$y_{,ij} = I g_{ij},$$

where the comma denotes covariant differentiation with respect to g_{ij} and I is an unspecified scalar. The cases $(n>1,\ m-n=1)$, $(n>1,\ m-n>1)$ and the cases $\rho=1$, $m-n=1$; $\rho=1$, $m-n>1$, as well as the cases $\rho=1$, $\sigma=1$, are treated separately. One of the main results is the establishment of the necessary and sufficient conditions that a tensor ${}^a g_{\alpha\beta}$ of type $(n>1,\ m-n>1)$ may represent an E_m. Related to this is the theorem for such a case that either each of the component tensors represent E's or S's (spaces of constant curvature), or $n=m-n=2$ and neither of them represents an S_2. Among the particular cases treated are S_m with conformally separable fundamental tensor, and E_4 with tensors of the type (2, 2). One of the properties concerning manifolds of type $(n>1,\ m-n=1)$ is that, if an E_m admits a one parameter family of totally umbilical hypersurfaces, then they are conformal to one another and each of them has constant scalar curvature. *D. J. Struik.*

Wong, Yung-Chow. Family of totally umbilical hypersurfaces in an Einstein space. Ann. of Math. (2) **44**, 271–297 (1943). [MF 8285]

In the first part of the paper necessary and sufficient conditions are given that a one parameter family of V_n with given fundamental tensor can be imbedded in an Einstein space E_{n+1} as totally umbilical hypersurfaces which are not totally geodesic. Several special cases are considered. In the second part some theorems are derived concerning conformal-Euclidean spaces C_n. This requires the detailed study of the equation

$$\rho_{,ij} + \varphi L_{ij} = \omega g_{ij}, \qquad i, j = 1, 2, \cdots, n,$$

where ω is an unspecified scalar,

$$L_{ij} = R_{ij} - (R/2(n-1)) g_{ij},$$

R_{ij} the contracted curvature tensor, and $\varphi = \varphi(\rho)$ is a given function of ρ such that $\varphi' \neq -1/(n-2)$. It is shown that for a C_n (not of constant curvature) this equation has a solution for $\rho = \rho(\bar{\rho})$ if and only if for the C_n the tensor L_{ij} has either the form

$$L_{ij} = \bar{\eta}(\bar{\rho}) \bar{\rho}_{,\bar{i}} \bar{\rho}_{,\bar{j}} \neq \bar{\xi}(\bar{\rho}) g_{ij}, \qquad g_{\bar{i}\bar{j}} \bar{\rho}_{,\bar{i}} \bar{\rho}_{,\bar{j}} \neq 0,$$

or

$$L_{ij} = \bar{\eta}(\bar{\rho}) \bar{\rho}_{,\bar{i}} \bar{\rho}_{,\bar{j}}, \qquad g^{\bar{i}\bar{j}} \bar{\rho}_{,\bar{i}} \bar{\rho}_{,\bar{j}} = 0.$$

Canonical forms for the fundamental tensor of such C_n are obtained. There is an incidental relation with the subprojective spaces of Kagan. Several of the theorems obtained are generalizations of results of Brinkmann, Fialkow, Yano and of Wong himself. *D. J. Struik* (Cambridge, Mass.).

Wong, Yung-Chow. A note on complementary subspaces in a Riemannian space. Bull. Amer. Math. Soc. **49**, 120–125 (1943). [MF 7992]

If the fundamental form $g_{\lambda\kappa} dx^\lambda dx^\kappa$, $\kappa, \lambda = 1, 2, \cdots, n$, of a Riemannian V_n can be written in the form

$$F = g_{ab} dx^a dx^b + g_{qp} dx^q dx^p,$$

where $a, b, c, \cdots = 1, \cdots, m$; $p, q, \cdots = m+1, \cdots, n$, and g_{ab} and g_{qp} may contain all x^ϵ, then the families of $V_m : x^p = \text{const.}$ and $V_{n-m} : x^a = \text{const.}$ are called complementary subspaces. This paper gives several properties of the normal complexes of these subspaces, especially for the case

that a vector v^σ exists such that the equation $v_\sigma H_{ab}^{\cdots\sigma} = {}'g_{ab}$ is satisfied at every point of V_m, where ${}'g_{ab}$ is the fundamental tensor of the V_m. The V_m, in this case, is called totally semi-umbilical. One of the theorems established in this paper is as follows. In a V_n with fundamental form F the normal complexes of the subspaces $x^p = \text{const.}$ are of dimension $m_1, m_2, \cdots$ if and only if the matrices

$$(\partial_p g_{ab}), \quad \begin{pmatrix} \partial_p g_{ab} \\ \partial_p \partial_f g_{ad} \end{pmatrix}$$

are of ranks $m_1, m_1+m_2, \cdots$, respectively. *D. J. Struik.*

Shabbar, Mohammad. One parameter groups of deformations in Riemannian spaces. J. Indian Math. Soc. (N.S.) 6, 186–191 (1942). [MF 8378]

The increment of any tensor in Riemannian space (or any space in which covariant differentiation is defined) under an infinitesimal deformation $u^i(x)$ is exhibited in explicit tensor form. The effect on a tensor or connection of the finite transformations generated by u^i is then given as a power series in the iterated increments. The formalism permits simplified proofs of various well-known theorems. At the same time conditions that certain spaces admit a group of motions take improved form. *J. L. Vanderslice.*

Galvani, Octave. Sur les connexions euclidiennes à courbure non nulle réalisables par des congruences de droites. C. R. Acad. Sci. Paris 214, 733–735 (1942). [MF 7884]

A continuation of the note in the same C. R. 214, 337–339 (1942); cf. these Rev. 4, 171. *T. Y. Thomas.*

Chern, Shiing-shen. On the Euclidean connections in a Finsler space. Proc. Nat. Acad. Sci. U. S. A. 29, 33–37 (1943). [MF 7680]

The different Euclidean connections in a Finsler space defined by other authors (Cartan, Berwald, Synge, Taylor) are shown to be special elements of an infinite set of Euclidean connections. The equivalence of any two of these Euclidean connections for two different Finsler spaces (under point transformations) is necessary and sufficient for the equivalence of the Finsler spaces themselves. To obtain these results a set of invariants is constructed for an arbitrary Finsler space, which is complete in the sense that it determines the space up to a point transformation. Then the Euclidean connections are expressed in terms of these invariants. *H. Busemann* (Chicago, Ill.).

Chern, Shiing-shen. A generalization of the projective geometry of linear spaces. Proc. Nat. Acad. Sci. U. S. A. 29, 38–43 (1943). [MF 7681]

The r-dimensional linear subspaces of an n-dimensional projective space form, in an obvious way, an N-dimensional space, $N=(r+1)(n-1)$, with a projective connection. It is shown that in any N parameter family of r-dimensional varieties in an n-dimensional space a projective connection can be defined, and in such fashion that it coincides for $r=1$ with the geometry of paths. The essential step is the construction of a set of $n(n+2)$ linearly independent Pfaffian forms in $n(n+2)$ auxiliary variables, where the forms are invariant under general transformations of these variables. *H. Busemann* (Chicago, Ill.).

MECHANICS

Astronomy

Rabe, E. Bemerkungen über die oskulierenden Elemente in der speziellen Störungsrechnung. Astr. Nachr. 272, 208–211 (1942). [MF 8569]

The author presents arguments against a remark by H. Samter [Enzykl. Math. Wiss., vol. 6, part 2, no. 19, 1922, p. 961, footnote] opposing the validity of Kepler's third law for osculating elements. Samter has withdrawn his statement [Astr. Nachr. 246, 407 (1932)]. *W. Kaplan.*

Okyay Kabakcioğlu, Tevfik. Verallgemeinerung und Anwendung der Wilkens'schen Theorie im Problem der mehrfachen Kommensurabilitäten. Rev. Fac. Sci. Univ. Istanbul. Ser. A. 6, 192–223 (1941). (German. Turkish summary) [MF 8454]

The problem considered is that of the perturbations of an asteroid by Jupiter and Saturn when their respective mean motions n, n', n'' satisfy a commensurability condition $pn+qn'+rn''=0$. This generalization of the classical resonance problem of the solar system [see Brown and Shook, Planetary Theory, Cambridge University Press, 1933, chap. 8] was studied by Wilkens [S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1, 71–101 (1933)]; in the case $p=1$, $q=3$, $r=2$ he obtained a differential equation for the quantity $K=pl-ql'+rl''$, where l, l', l'' are the longitudes, and by solving it found expressions for the other elements of the orbit of the asteroid. Wilkens, however, neglected all terms of first or higher degree in the eccentricities. The present paper considers the case $p=2$, $q=5$, $n=3$ and takes into account terms of first degree in the eccentricities. A differential equation for the variable K is again found and integrated in special cases, the results being similar to that for the pendulum equation. It is then indicated how the other elements of the orbit may be computed. *W. Kaplan* (Ann Arbor, Mich.).

Zumkley, J. Ein numerisch gerechneter Spezialfall des allgemeinen Dreikörperproblems in vereinfachter Behandlung. Astr. Nachr. 272, 66–76 (1941). [MF 8574]

The author solves, by numerical integration, the following special case of the three-body problem: the masses m_1, m_2, m_3 are equal and move in the xy-plane; for $t=0$ their coordinates (x, y) are respectively $(2, 0)$, $(1, 0)$, $(3.5, 0)$ and their velocities (x', y') are $(0, 0)$, $(0, -1, 5)$, $(0, 1)$. For the computation the motion is referred to relative coordinates r_3-r_2, r_1-r_3, r_2-r_1, where r_1, r_2, r_3 are the position vectors. The solution is tabulated and then graphed in various coordinate systems. As t becomes large m_3 recedes indefinitely, asymptotic to a hyperbola, while m_1 and m_2 move about their common center of mass in paths approaching ellipses. Replacing t by $-t$ reflects the solution in the y-axis. *W. Kaplan* (Ann Arbor, Mich.).

Tüzemen, Edibe. Généralisation d'un théorème de W. Gleissberg. Rev. Fac. Sci. Univ. Istanbul. Ser. A. 6, 237–243 (1941). (French. Turkish summary) [MF 8455]

The author proves the theorem: in any star in mechanical equilibrium such that $\bar{\rho}(r)$, the mean density of the mass M_r at a distance less than r from the center, is a decreasing function of r, the expression $(3n+2)(\bar{P}^{(n)}-P)$ is a decreasing function of the continuous real variable n $(n>0)$ and

has as limit, for $n\to\infty$, the value L, where

$$L=G(\tfrac{4}{3}\pi)^{1/3}\bar{\rho}^{4/3}M_r^{2/3}.$$

Here P is the pressure at distance r, and $\bar{P}^{(n)}$ is a mean pressure defined by

$$\bar{P}^{(n)}=(1/M_r^n)\int_0^{M_r^n}Pd(M_r^n);$$

G is the constant of gravitation. Gleissberg [Monthly Not. Roy. Astr. Soc. **99**, 188–195 (1939)] had proved that for n integral the sequence $(3n+2)(\bar{P}^{(n)}-P)$ is decreasing as n increases and has the value L as a lower bound. Similar theorems have been proved by Eddington [Internal Constitution of the Stars, Cambridge University Press, 1926, p. 91], Milne [Monthly Not. Roy. Astr. Soc. **96**, 179–184 (1936)] and Chandrasekhar [Monthly Not. Roy. Astr. Soc. **96**, 644–647 (1936)]. *W. Kaplan* (Ann Arbor, Mich.).

Hnatek, A. Über die Ermittlung der Randverdunkelung bei Bedeckungsveränderlichen. II. Astr. Nachr. **272**, 159–165 (1942). [MF 8572]

The first part appeared in Astr. Nachr. **261**, 361 (1937).

Chandrasekhar, S. Dynamical friction. I. General considerations: the coefficient of dynamical friction. Astrophys. J. **97**, 255–262 (1943). [MF 8231]

Chandrasekhar, S. Dynamical friction. II. The rate of escape of stars from clusters and the evidence for the operation of dynamical friction. Astrophys. J. **97**, 263–273 (1943). [MF 8232]

In the first part it is pointed out that, if the changes in velocity of a star are to be considered as due to a random fluctuation in the force upon it, then an initial Maxwell distribution of velocities would disintegrate as the time becomes large. In order to obtain a force distribution which leaves the Maxwell distribution invariant, a "dynamical friction" term, which gives a deceleration proportional to the velocity in the direction of motion, is added to the random force [cf. the modern theory of Brownian motion as presented, for example, in S. Chandrasekhar, Rev. Modern Phys. **15**, 1–89 (1943), in particular, pp. 20–42; these Rev. **4**, 248]. It is then shown that an analysis of the actual fluctuations in the forces acting on stars due to double encounters, using the theory developed by S. Chandrasekhar [Principles of Stellar Dynamics, University of Chicago Press, Chicago, Ill., 1942, ch. 2; these Rev. **4**, 57], leads to fluctuations in velocities which prove (granting certain approximations) the presence of dynamical friction; an explicit expression for the coefficient of dynamical friction is obtained.

The problem in the second part is to predict the rate of escape of stars from a galactic cluster such as the Pleiades. It is shown that one can deduce the expectation that a star will reach the critical velocity of escape from the probability function W which gives the velocity distribution at any time t. If the velocity fluctuations are random, W satisfies a partial differential equation of heat-conduction type; if dynamic friction is taken into account, W satisfies a more general partial differential equation. It is shown that, in the former case, the expectation would be too high to be consistent with experimental evidence for the life of the clusters; but if dynamic friction is taken into account, the expectation becomes of the correct order of magnitude. The determination of the function W in the latter case requires solution of a boundary value problem related to that for the Hermite polynomials. *W. Kaplan* (Ann Arbor, Mich.).

Hydrodynamics, Aerodynamics

Sauer, R. Charakteristikenverfahren für die eindimensionale instationäre Gasströmung. Ing.-Arch. **13**, 79–89 (1942). [MF 8579]

The unsteady one-dimensional flow of a compressible fluid leads to the differential equation

$$(c^2-u^2)\varphi_{xx}-2u\varphi_{xt}-\varphi_{tt}=0,$$

where φ denotes the velocity potential, $u=\varphi_x$ the velocity of the fluid and c the velocity of sound. By the Legendre transformation $u=\varphi_x$, $q=\varphi_t$, $\Phi=xu+tq-\varphi$, this differential equation is transformed into

$$(c^2-u^2)\Phi_{qq}+2u\Phi_{uq}-\Phi_{qq}=0.$$

The characteristics of the first equation are given by $dx/dt=u\pm c$ ("space-time characteristics"), those of the second equation by $dq/du=-(u\pm c)$ ("state characteristics"). Since the velocity of sound can be expressed in terms of u and q, the equation of the state characteristics can be integrated independently of the conditions of the particular flow under consideration. The state characteristics are drawn in the u, q plane. Together with the initial conditions and the boundary conditions of the problem, the orthogonality relation

$$\frac{dt}{dx}\cdot\frac{dq}{dt}=-1$$

permits then the drawing of the space-time characteristics in the x, t plane. The graphical procedure is essentially the same as that of the well-known method which A. Busemann and L. Prandtl gave for the steady two-dimensional flow of a compressible fluid. The following examples are discussed: (1) piston moving in an infinite cylinder, (2) piston moving in a finite cylinder which is closed at the other end, (3) two pistons moving in opposite directions in a cylinder open at both ends, (4) spreading of an initial disturbance in an infinite cylinder (Riemann's problem). The method is applicable in the subsonic as well as in the supersonic case. In the present paper only continuous motions are discussed; if the conditions of a problem lead eventually to the formation of shock waves, the motion is studied only up to the instant when the first shock wave appears. *W. Prager.*

Bers, Lipman. Concerning the acceleration potential. Quart. Appl. Math. **1**, 93–96 (1943). [MF 8193]

The author derives the condition $\Delta p=0$, on which assumption the acceleration potential is usually based, for the steady incompressible potential flow around a thin airfoil of infinite span. The derivation involves a sequence of approximations; the author conjectures that quantitative estimates of the error in each of these approximations are possible, but he does not actually obtain such estimates. The central approximation is the replacement of the two singularities at the (sharp) leading edge L and the stagnation point S by a single doublet at L. This replacement is performed in a certain auxiliary plane which is approximately the plane in which the exterior of the unit circle is the image of the exterior of the airfoil. *P. W. Ketchum.*

Unna, P. J. H. Theory of sea waves. Nature **151**, 479–480 (1943). [MF 8336]

The author reviews the theoretical aspect of the problem of sea waves for the case where the ratio of the height H of the wave to the wave length L is not small. These "steep" waves are not any more trochoidal and, for example, the

wave velocity V will depend on the steepness ratio $S=H/L$. The author discusses results obtained by Rayleigh, Airy, Lamb and others regarding the necessary corrections in the theory for large S. The discussion includes the case of waves in shoaling water, especially the transitional range where the waves pass from the limiting case of "deep water," where $V=(gL/2\pi)^{\frac{1}{2}}$, to shallow water, where in the limit $V=(gD)^{\frac{1}{2}}$; D denotes the depth of the water.

H. W. Liepmann (Pasadena, Calif.).

Poritsky, H. Vortices in fluid flow. J. Math. Phys. Mass. Inst. Tech. 21, 218–239 (1942). [MF 8170]

This is an expository paper concerning the rôle of vortices in the theory of fluid (and gaseous) flow. The writer does not attempt to give a detailed account of the mathematical theory but presents only enough equations to be able to give mathematical definitions of the concepts involved.

The paper begins with the fundamental equations of hydrodynamics including Bernoulli's equation. Everyday examples of vortices are given and the vorticity vector and the circulation around a closed curve are then defined mathematically. Rotational and irrotational flow are defined, the elementary consequences of irrotational flow are discussed and some examples of both types of flow are given. In particular, Kelvin's theorem for ideal fluids on the constancy with time of the circulation around a closed curve (as it moves) is stated, from which it follows that irrotational flow must always remain irrotational and that no vortices can develop. It is then pointed out that there would be no net force on an infinite cylinder (of any shaped cross section) unless it was assumed that curves encircling the cylinder had a nonzero circulation (that is, that there was a vortex within the cylinder). The ideal fluid theory can be made to account rather accurately for the "lift" on an airfoil by making such an assumption. If we consider a body of finite dimensions moving in ordinary space, then, if the fluid is perfect and the flow is assumed smooth behind the body, it follows that the fluid can exert no force on the body. This difficulty may be got around even in an ideal fluid by assuming that an initial vortex formed behind the body at the beginning of the motion and that there are vortex lines which are simple closed curves which pass through the object and this initial vortex, stretching out as the body proceeds. The effect of taking account of viscosity is discussed and other ways in which vortices may be developed are pointed out. Various types of actual flows are illustrated by pictures and diagrams. *C. B. Morrey, Jr.*

***Lin, C. C. On the Motion of Vortices in Two Dimensions.** University of Toronto Studies, Applied Mathematics Series, no. 5. University of Toronto Press, Toronto, Ont., 1943. 39 pp. $1.00.

The author considers the two dimensional motion of an incompressible fluid in a region R bounded internally by a number of fixed closed boundaries C_k and: (a) bounded externally by a closed curve C_0 or (b) unbounded externally or (c) limited by curves C_0 extending to infinity. The motion of the fluid is irrotational except for the existence of n free point vortices of strengths k_i at points $P_i(x_i, y_i)$. The author's problems are (a) to establish the existence of the "Kirchhoff-Routh path function" W, such that the motion of the ith vortex is given by

$$k_i\frac{dx_i}{dt}=-\frac{\partial W}{\partial y_i},\quad k_i\frac{dy_i}{dt}=\frac{\partial W}{\partial x_i},$$

(b) to determine an explicit formula for W in terms of a generalized Green's function of the region, (c) to study the behavior of W under a conformal transformation. It is shown that, if W exists, then n "Routh stream functions" $\chi_{(i)}$ exist such that the motion of the ith vortex is given by

$$\frac{dx_i}{dt}=-\frac{\partial\chi_{(i)}}{\partial y_i},\quad \frac{dy_i}{dt}=\frac{\partial\chi_{(i)}}{\partial x_i}.$$

In the first section, a historical survey of the problem is given. In the second section the author assumes (for the present) the existence of a generalized Green's function $G(x, y; x_i, y_i)$ which: (a) is harmonic at all interior points of R except (x_i, y_i); (b) has a logarithmic singularity at (x_i, y_i); (c) has a specified value over the interior boundaries; (d) satisfies certain regularity conditions over the exterior boundaries. Four fundamental theorems are proved. Theorem I assumes the existence of the normal derivative of G on the boundaries and proves the uniqueness and the reciprocity property of this function. Further, it is shown that the function

$$g(x, y; x_i, y_i)\equiv G(x, y; x_i, y_i)-\frac{1}{2\pi}\log r_i,$$

where r_i is the Euclidean distance from (x_i, y_i) to (x, y), is harmonic throughout R, possesses the reciprocity property and satisfies the relation

$$\frac{\partial}{\partial x_i}g(x_i, y_i; x_i, y_i)=2\lim_{P\to P_i}\frac{\partial}{\partial x}g(x, y; x_i, y_i),$$

as well as a similar relation for $\partial g/\partial y_i$. Theorem II shows that the stream function $\psi(x, y; x_1, y_1; \cdots; x_n, y_n)$ for the motion in R of an incompressible fluid in which n vortices of strengths k_i are situated can be written

$$\psi(x, y; x_1, y_1, \cdots; x_n, y_n)=\psi_0(x, y)+\sum k_iG(x, y; x_i, y_i),$$

where ψ_0 is the stream function for a potential flow in R (it represents the flow due to all causes except the vortices). Theorem III gives an explicit formula for W in terms of the generalized Green's function of R, that is, for steady motion at infinity. In theorem IV it is shown that, under a conformal transformation, W becomes

$$\tilde{W}=W+\sum\frac{k_i^2}{4\pi}\log\left|\frac{dz}{d\bar{z}}\right|_{P_i}.$$

The third section is devoted to: (a) a generalization of W to the case where the motion is not steady and bound vortices exist; (b) a physical interpretation of W as the "interaction energy" of the system; (c) a study of the formulas for $G(x, y; x_i, y_i)$ in the cases where the region R is unbounded or simply connected or contains a single vortex. For this last problem, a rule is given for finding the Routh stream function. Application is made to a special problem previously solved by Walton [Proc. Roy. Irish Acad. Sect. A. 38, 29–39 (1928)] who used other methods [the number 4.17 on p. 33, line 3 should read 8.17].

Finally, the author turns to the question of the existence of the generalized Green's function $G(x, y; x_i, y_i)$. It is shown that, according to the nature of the external boundary, two types of functions $G(x, y; x_i, y_i)$ exist. By identifying the functions $G(x, y; x_i, y_i)$ with the potential functions whose unique existence has been proved by Koebe [Acta Math. 41, 306–344 (1918)] the existence of the author's generalized Green's functions is proved. Further, it is shown that, under the author's additional requirements as

to the nature of $G(x, y; x_i, y_i)$, this function will possess the reciprocity property. Here, the existence of the normal derivative of $G(x, y; x_i, y_i)$ is not required. *N. Coburn.*

Ertel, Hans. Über hydrodynamische Wirbelsätze. Phys. Z. **43**, 526–529 (1942). [MF 8620]

The equations for the individual change of vorticity with time in an ideal compressible fluid are generalized by using a rotating Cartesian coordinate system and by multiplying them scalarly with a hydrodynamic field function. In the customary equations the vorticity changes are related to the size of the area bounded by two isobaric and two isosteric surfaces which differ by one unit of pressure and specific volume, respectively. The new equations represent essentially a relation between the changes with time of the product of the field function in the vorticity and the volume of the unit cells bounded by the isobaric and isosteric surfaces and by the surfaces of the field function.

B. Haurwitz (Cambridge, Mass.).

Shvetz, M. The course of the velocity of wind in twenty-four hours and the turbulent diffusion. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **1943**, 55–59 (1943). (Russian. English summary) [MF 8717]

In this article there are simplified the equations of motion for the turbulent atmosphere. It is shown that the course of the velocity of wind in twenty-four hours is conditioned by the course of turbulent diffusion in twenty-four hours, the change of the coefficient of diffusion with the height being of essential importance. The calculated course of the velocity of wind in twenty-four hours is compared with the observed one. The results of comparison are entirely satisfactory. *Author's summary.*

Keulegan, Garbis H. and Patterson, George W. Effect of turbulence and channel slope on translation waves. J. Research Nat. Bur. Standards **30**, 461–512 (1943). [MF 8458]

As the theory of unsteady flow in a channel is based here upon the theory of Reynolds stresses, the Reynolds equations of mean motion are first developed but the Reynolds stresses R_{xx}, R_{xy}, etc., are not expressed in terms of the mean velocity gradients as in the work of Boussinesq, since his assumptions regarding the value of the eddy viscosity seem invalid in the light of present experimental data. In the next part of this section the equation of continuity for mean flow in channels is given.

A first-order theory of wave-propagation is given in section III, beginning with the equation of mean flow in prismatic channels. The final equation involves the hydraulic radius of the channel and one coefficient of velocity distribution, namely, that relating to the squares of velocities. Use is made also of a channel coefficient of friction which can be evaluated when certain simple assumptions are made. When initially there is uniform flow in a very wide rectangular channel the equation of propagation of waves takes the form

$$\partial^2 h/\partial t^2+2U_0\partial^2 h/\partial x\partial t+(U_0^2-gH_0)\partial^2 h/\partial x^2=0$$

[there is a misprint in the paper, U_0^2 being written for U_0 in the second term] and a solution of type $h=F(x-w_0t)$ is considered. The height H of the free surface is given by $H(x, t)=H_0+h(x, t)$ and the velocity U by $U(x, t)=U_0+u(x, t)$. It is found that $u(x, t)$ can be derived from $h(x, t)$ by a simple equation. The relation to former work on the motion of irrotational waves in still water is pointed out. The physical significance of the mathematical assumptions is carefully discussed and a workable criterion is found for the applicability of the theory. Diagrams are given to show the limitations on the height of short waves of ascending and descending types which can be propagated without change of form, the two types being distinguished by the inequalities $U_0>w_0$, $U_0<w_0$, respectively.

The next section deals with the propagation of discontinuities of slope. Jumps in the values of the space and time derivatives of H and U are considered. In a descending wave the velocity of the wave front is found to be $U_0+(gH_0)^{\frac{1}{2}}$. The analysis does not give the magnitude of the jump. The effect of velocity distribution on the celerity of short waves is next considered and again a kind of wave represented by $h=F(x-w_0t)$ is determined. The results are discussed with the aid of numerical data of Nikuradse and Bazin.

The next section deals with a second-order theory of wave propagation. The equation of mean motion is now given for the case in which the vertical acceleration is appreciable, the case of a wide rectangular channel is adopted as before and the horizontal velocities in a section are treated as practically uniform. The equation for h is still of the second order in t but is of the fourth order in x. The idea of the celerity of a volume element, introduced by Boussinesq, is found to be useful. The wave profile is found to be deformed in a case in which there is a marked simplification of the differential equation. The effect of friction and of slope of bed are discussed and the deformation of a straight sloping front is calculated. The change of height of an abrupt wave front is also computed. An interesting problem, which can be simply treated with the aid of a method due to Favre, is that of intumescences produced by operating locks in a canal carrying water. The waves produced are small in height and so long that curvature may be neglected. Use can then be made of Boussinesq's solution of the wave equation when slope and friction are not negligible. The case of an ordinary canal containing still water before disturbances are set up is solved with the aid of another integral of Boussinesq suitable for a frictional bed without slope. Criteria are given under which these solutions of Boussinesq are applicable. *H. Bateman.*

Reichardt, H. On a new theory of free turbulence. J. Royal Aeronaut. Soc. **47**, 167–176 (1943). [MF 8652]

The author proposes the inductive method to develop a theory of free turbulence, that is, to start with measured quantities and to try to find from them the governing laws of turbulent exchange. New careful measurements of the total head distribution in jets and wakes and on a free boundary have been made for this purpose. The author shows that his measured profiles follow closely the error function (jet and wake) or the error integral function (free boundary). Based on this fact the equation

$$\frac{\partial\overline{u^2}}{\partial x}=\Lambda\frac{\partial^2\overline{u^2}}{\partial y^2}, \qquad \Lambda=-\frac{b\,db}{2\,dx},$$

is proposed as the expression of a "momentum diffusion law." Here u denotes the velocity in the direction x, Λ is some "conductivity coefficient" and b is the width of the turbulent mixing region. The sinusoidal velocity distribution in the wake of a grid can also be obtained from the above equation but with a different expression for Λ. Quantities like the normal velocity component v, the turbulent

shearing stress, etc. can then be computed from the Stokes-Navier equation. [Reviewer's remark: It appears that for an attempt to apply the inductive method to a problem so complex as turbulence measurements of the mean velocity distribution alone are not sufficient.] *H. W. Liepmann.*

Dryden, Hugh L. A review of the statistical theory of turbulence. Quart. Appl. Math. **1**, 7–42 (1943). [MF 8185]

This is an expository review of the present state of the statistical theory of turbulence inaugurated by G. I. Taylor in 1935 and developed by Taylor, von Kármán, the present author and others. The introductory paragraphs are devoted to definitions and statement of the principal assumptions. The main part of the paper is concerned with isotropic turbulence, namely, decay of isotropic turbulence, effect of contraction, the correlation tensor function, correlation between derivatives of the velocity fluctuation, triple correlations, propagation of correlation with time, self-preserving correlation functions and diffusion in isotropic turbulence. The spectrum of turbulence is discussed and its relationship to the correlation is shown. The paper closes with brief discussions of the statistical theory of nonisotropic turbulence, diffusion is nonisotropic turbulence and correlation in turbulent flow through a pipe. The author makes a practice of examining the conclusions of the theory in the light of available experimental evidence. The review, therefore, affords a rather complete picture of the present state of experimental research relating to the statistical theories. *W. R. Sears* (Inglewood, Calif.).

Synge, J. L. and Lin, C. C. On a statistical model of isotropic turbulence. Trans. Roy. Soc. Canada. Sect. III. (3) **37**, 1–35 (1943). [MF 8457]

An attempt to develop a theory of homogeneous isotropic turbulence based on a statistical model is presented. The authors assume the turbulent field to be formed by an assembly of statistically distributed Hill vortices, that is, spherical vortices with a finite radius a and translational velocity U. The velocity field at a distance r from the center of a Hill vortex written in tensor notation with l_j denoting the direction cosines of the direction of U is given by:

$$\text{(1a)} \qquad u_i=\left\{\frac{3}{2}\frac{Ua^3}{r^5}r_ir_j-\frac{1}{2}\frac{Ua^3}{r^3}\delta_{ij}\right\}l_j, \qquad r\geqq a,$$

$$\text{(1b)} \qquad u_i=\left\{\frac{3}{2}\frac{U}{a^2}r_ir_j+U\left(\frac{5}{2}-\frac{3r^2}{a^2}\right)\delta_{ij}\right\}l_j, \qquad r\leqq a.$$

The main task of the theory is now to compute the correlation tensors of various order defined by von Kármán and Howarth. In order to compute these average values assumptions regarding the distribution function of the statistical elements have to be made. The authors assume in analogy with the kinetic theory of gases that in first approximation the vortices behave independently of each other. To compute the correlation functions the authors proceed in two steps. (i) All vortices are assumed to be of a single "type," that is, have the same radius a and velocity U. (ii) The vortices are statistically distributed with respect to type. In the latter case an additional assumption regarding the distribution function with respect to U and a has to be made. If $H(a)$ defines the distribution of energy among the sizes of vortices given by a, the authors assume

$$H(a)=Aa^2e^{-a/a_0}$$

with constant A and a_0. The correlation tensors can then be computed. Of main interest are the tensor of second order for which experimental data are available and the tensor of third order which physically gives the amount of energy diffusion in the turbulent field. The second order correlation functions $f(\xi)$ and $g(\xi)$ satisfy von Kármán's relation

$$f-g=-\tfrac{1}{2}\xi\frac{df}{d\xi}$$

and $f(\xi)$ agrees well with experimental data if vortices of different type are considered (case (ii)). The triple correlation and in fact all correlations of odd number are found to vanish. This latter fact is an unsatisfactory result of the theory and as the authors point out due to the over simplification of the model, namely the fore and aft symmetry of the Hill vortex. A further difficulty of the model arises from the fact that the derivative of the induced velocities (1a) and (1b) are discontinuous at $r=a$. Hence, viscous forces would produce discontinuous shear on the surface of the vortex and thus a theory of the decay of turbulence based on this model is not possible. *H. W. Liepmann.*

Theory of Elasticity

Sadowsky, M. A. A principle of maximum plastic resistance. J. Appl. Mech. **10**, A-65–A-68 (1943). [MF 8424]

The following heuristic principle is established. "Among all statically possible stress distributions (satisfying all three equations of equilibrium, the condition of plasticity and boundary conditions), the actual stress distribution in plastic flow requires a maximum value of external effort necessary to maintain flow." In the case of combined torsion and tension of a prismatic bar, this principle is interpreted as requiring a stress distribution which furnishes a maximum value of the axial force for a given value of the torque. A similar interpretation is given in the case of a tube subjected to torsion, tension and internal pressure: the stress distribution is determined so as to furnish a maximum value of the axial force for given values of the torque and the internal pressure. In order to justify the principle, the author applies it to the two problems mentioned above. In the case of combined torsion and tension the result is identical with that which A. Nadai obtained by means of a complete analysis of the stresses and strains involved. [As applied to this case, the principle can easily be deduced from the equations furnished by this analysis.]

W. Prager (Providence, R. I.).

Reissner, Eric. On the calculation of three-dimensional corrections for the two dimensional theory of plane stress. Proc. Eastern Photoelasticity Conference no. **15**, 23–31 (1942). [MF 8340]

The results of the theory of plane stress may be seriously in error in regions in which appreciable changes of stress occur over distances of the same order of magnitude as the thickness of the plate-shaped body to which the theory is applied. The paper presents a method for obtaining approximate three-dimensional corrections for the usual two-dimensional theory. A system of partial differential equations (with two independent variables) for the functions representing these corrections are obtained by the use of the minimum energy principle. The author states that this system can be solved for the cases of a plate bounded by

two parallel planes, that is, an infinite strip, and of a plate bounded by two concentric cylinders including the case of a circular hole in an infinite sheet. *H. W. March.*

Grosser, C. E. Determination of the sum of principal stresses by graphical and mechanical means. Proc. Eastern Photoelasticity Conference no. 15, 43–53 (1942). [MF 8341]

It is shown that for a "shear membrane," in which the only internal forces are shearing forces normal to the membrane, the displacement perpendicular to the membrane obeys Laplace's equation in two dimensions. Such a membrane can be approximated by a grid of square prisms which slide freely against each other except that each prism is connected with the four adjacent prisms by coil springs whose axes are perpendicular to the membrane. The displacement of any block will thus be the average of the displacements of the four adjacent blocks. By fixing the displacements of the blocks along a given boundary curve, Dirichlet's problem may be solved approximately. In particular, Green's function G can be found by displacing one of the blocks inside the boundary by a given amount and holding the blocks on the boundary at zero displacement. The shearing stress at the boundary of the tent-like surface thus obtained is proportional to the normal derivative $\partial G/\partial n$ at the boundary. If this surface is approximated by fitting side by side a number of small sectors from the (known) Green's function for circles of differing radii, and the shear between the sectors is neglected, there is obtained a simple expression for the shearing stress along the boundary. This expression may be substituted for $\partial G/\partial n$ in the usual integral for the solution of the Dirichlet problem in terms of Green's function, and the integral evaluated numerically. There is obtained in this way a simple numerical procedure for solving the Dirichlet problem, which is claimed to be fairly accurate if the boundary is not too irregular. A model shear membrane consisting of 81 elements or blocks was also constructed. *P. W. Ketchum.*

Glagolev, N. I. Elastic stresses along the bottom of a dam. C. R. (Doklady) Acad. Sci. URSS (N.S.) **34**, 187–191 (1942). [MF 7602]

The problem of a rigid dam on an elastic foundation is treated as a problem in plane stress. It is assumed that friction is acting along the line of contact between dam and foundation. This complicates matters mathematically as compared with the assumption of no shear along the line of contact. The complex integral equation of the mixed boundary problem, which is obtained with the help of results of Muschelišvili, can be solved explicitly, as it is a special case of the equation

$$a(x)u(x)-\lambda \oint_{-1}^{1} u(y)dy/(y-x)=f(x),$$

for which T. Carleman [Ark. Mat. Astr. Fys. 16 (1922)] has given the solution. The author's formulas generalize results previously obtained by different methods by V. M. Abramov [C. R. (Doklady) Acad. Sci. URSS (N.S.) 17, 173–178 (1937)]. *E. Reissner.*

Philippow, A. P. Ein unendlich langer Balken, auf elastischem Halbraume liegend. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 6, 169–186 (1942). (Russian. German summary) [MF 7747]

The problem of bending of an infinite beam on a three-dimensional elastic foundation is treated by Fourier integral methods. Analogous results for the same problem have previously been obtained by M. A. Biot [J. Appl. Mech. 4, A-1–A-7 (1937)]. The solution involves the function $\int_0^x K_0(u)du$, for which a six-place table is given for $0 \leqq x \leqq 0.60$ in steps of 0.02. *E. Reissner* (Cambridge, Mass.).

Schubert, Gerhard. Zur Frage der Druckverteilung unter elastisch gelagerten Tragwerken. Ing.-Arch. 13, 132–147 (1942). [MF 8586]

The paper is concerned with the following problem: a rigid body (foundation) is pressed against an elastic half-space (ground) by given forces; to find the stresses transmitted across the surface of contact. The following simplifying assumptions are introduced: (1) no shearing stresses are transmitted at the surface of contact; (2) the foundation is either a prism (cylinder) of infinite length or a body of revolution; in the second case the given loads are supposed to have a resultant acting along the axis of revolution. Integral representations of the pressure distribution under the foundation are developed and numerous examples are studied. *W. Prager* (Providence, R. I.).

Wolf, K. Nachtrag zu meinem Aufsatz in Bd. XII, S. 259, "Kreiszylindrische Behälter auf nachgiebiger Unterlage". Ing.-Arch. 13, 110 (1942). [MF 8583]

The paper appeared in Ing.-Arch. 12, 259–264 (1941); these Rev. 4, 180.

Gran Olsson, R. Elastische Knickung gerader Stäbe von exponentiell veränderlichem Querschnitt unter dem Einfluss ihres Eigengewichtes. Ing.-Arch. 13, 162–174 (1942). [MF 8589]

The differential equation of the problem is solvable in terms of confluent hypergeometric functions. The numerical evaluation is carried out for three cases for which the series for the functions of the first kind terminates after 1, 2, 3 terms, respectively, and for which the functions of the second kind are expressible in terms of the exponential integral and a finite number of exponential functions. *E. Reissner* (Cambridge, Mass.).

Gran Olsson, R. Unsymmetrische Biegung der Kreisplatte von quadratisch veränderlicher Steifigkeit. III. Ing.-Arch. 13, 147–154 (1942). [MF 8587]

It is shown that the deflection of a thin elastic plate may be expressed in plane polar coordinates ϕ, r as a series of products of trigonometric functions in ϕ and hypergeometric functions in r, provided the thickness t of the plate is given by the expression $t=(c_0+c_1r^2)^{\frac{1}{2}}$. A problem where the deflection is of rotational symmetry is evaluated numerically and to this end a short table of the hypergeometric functions $F(\alpha, \beta, \gamma; x)$ is given for (1) $\alpha=-\beta=\alpha_0=(0.35)^{\frac{1}{2}}$, $\gamma=1$, (2) $\alpha=1+\alpha_0$, $\beta=1-\alpha_0$, $\gamma=2$, (3) $\alpha=2+\alpha_0$, $\beta=2-\alpha$, $\gamma=3$. *E. Reissner* (Cambridge, Mass.).

Ohlig, R. Die achsensymmetrisch belastete dicke Kreisplatte. Ing.-Arch. 13, 155–162 (1942). [MF 8588]

The Lagrange-Kirchhoff theory of thin plates breaks down either when the dimension of the load region is comparable to the thickness of the plate or when the thickness of the plate is not small in comparison with the lateral dimension. The exact theory for thick plates is given in detail by S. Woinowsky-Krieger [Ing. Arch. 4, 203–226, 303–331 (1933)], who used the method of expanding the solution in a series of normal functions. The present author uses the same method to calculate the bending moments for a simply supported circular plate. *H. S. Tsien.*